Classical Solutions to Double Oscillator Field Theory

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Contents

Abstract

The present paper aims at studying the physics of the double oscillator field theory from a classical, non-perturbative perspective. In introducing the topic a short but to the point treatment of the physics of the quantum mechanics double oscillator is presented. A subsequent section of the paper summarizes the current state of research in the self-interacting scalar fields theory (with an emphasis on the specific physical systems generating spontaneous symmetry breaking). We further discuss in a non-perturbative framework a few classes of solutions to the double oscillator field theory. The focus is on analyzing the classes of upshots for the equation

$\partial_t^2\phi - \nabla^2\phi + m^2\phi - m^2 aSign(\phi) = J(r)$,

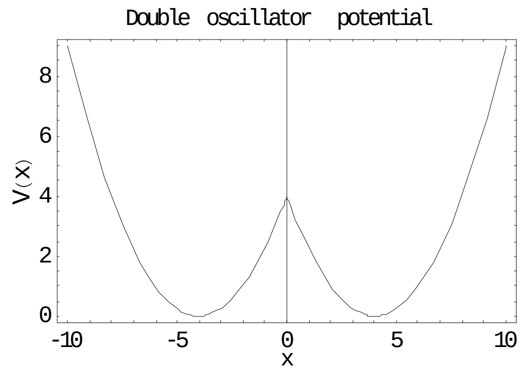
where *J(r)* is a source of the form $Q_0\delta(\vec{r})$. We particularize this problem, looking at solutions of the form $\phi = \eta(x) \pm a$. The classes of solutions are separately discussed in function of their complexity. Several results in this or in related domains are also acknowledged and further applied where possible. The paper introduces and leaves open the issue concerning similarity between the double oscillator field theory and the $\lambda\phi^4$ theory.

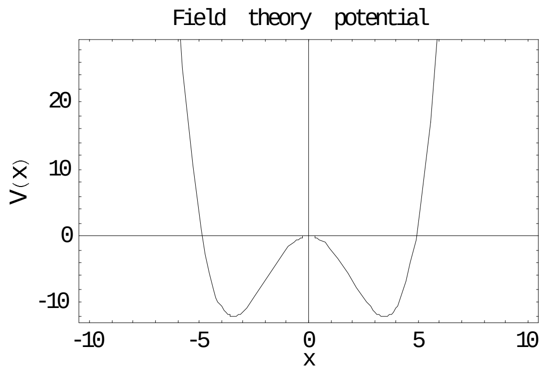
1. Introduction

We will start this paper by recalling that when we quantize the harmonic oscillator the creation operator evolves in a way that completely mimics the evolution of the classical solutions *[6]*. This is a settled result in physics and does not need detailed argumentation. The essence of the proof lies in the fact that since coherent states are built from the vacuum by hitting it with exponentiated creation operators, it's also true that coherent states evolve in a way which completely mimics the evolution of the corresponding classical solutions. We contend that there is no reason to think that such an argument would not apply as well for the quantum field theory. Consequently it is worth trying to find classical solutions to the double oscillator field theory, for instance.

Our target is to find classical solutions to the double well oscillator field theory, leaving a discussion open on the similarity between this theory and the theory of the self-interacting scalar fields ($\lambda\phi^4$). To illustrate this analogy, we will draw your attention to the explicit expression and behavior of the potentials in the case of the quantum mechanics double oscillator on the one hand and the $\lambda\phi^4$ field theory on the other hand. While in the first situation the potential will be of the form

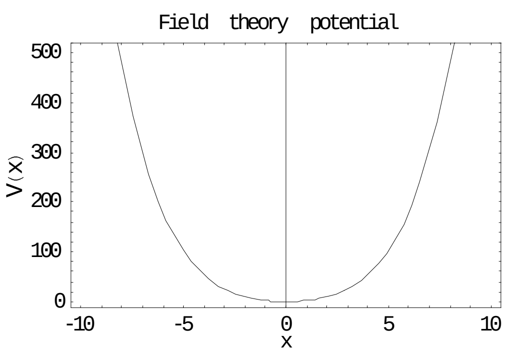
V$_{DO}$=$\frac{1}{2}m^2(|\phi|-a)^2$, in the latter case the potential will have the expression V$_{field}$=$\frac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4$. The corresponding graphs of these potentials will look (for particular values of the parameters, irrelevant as such for our purpose here) as follows:





We see immediately the similarity between these two plots, where the graph in the right is representative for negative values of m^2 (m=2i in this example). The meeting point is the substantive difference, where while the second of the potential functions is smooth enough (continuous and first order differentiable), the other one is just continuos but cannot be differentiated, thus it is not in the C^1 set of continuous and at least once differentiable functions.

It has to be kept in mind that in the case of positive values of m^2 for the second plot, we find ourselves in the realm of the self-interacting scalar fields where the symmetric vacuum is locally stable, as we will investigate in detail in the section devoted to the analysis of the self-interacting fields. The graph in this “natural case” will have the following layout (with m set to 0):



Summing up the findings herein, on an intuitive basis we would state that relations between the 2 theories are expected (or in a more mathematical formulation, their likelihood of being similar is considerable). On the one hand, the double well field theory (based on the quantum mechanics double oscillator in the last instance) will account for an *exactly solvable model* with spontaneous symmetry breaking. On the other hand, the general $\lambda\phi^4$ field theory is usually being approached by means of *perturbation theory or other approximations*[1]. Henceforth, a positive link between the two theories would be more than welcome. We do not make a purpose out of discussing in depth this issue, hereby limiting ourselves to classes of solutions for the double oscillator field theory. However, it is clear that the connection would constitute an interesting and challenging sequel to this paper.

If we were to summarize the scope of this paper in a few words, based on similarities between the models investigated we try to analyze specific instances within the physics of the phase-transition. Such instances will be regular charges (the trivial solutions), but also bubbles around charges. In this latter case the charges will act as condensation points around which the bubbles form, materializing thus the connection between the $\lambda\phi^4$ model with the double oscillator field theory model herein introduced. In the quantum field terminology, we are looking, *inter alia*, to bound-states of indefinite number of bosons, linked to such an interior charge (bubbles) or simply to the exterior, self-sustained, regular charges. Hence the classical solutions to the double field oscillator are elements of key importance in the phase transitions that can occur as soon as the potential has two minima (as spotted in the plots above).

In the next two sections the similarities and differences between the 2 field theory models will become more obvious, as a detailed analysis will be performed on each of them. The main chapters following afterwards are directly addressing the classical solutions to the double oscillator field theory.

2. Quantum mechanics of the double oscillator

Before starting our attempt to find classical solutions to the field theory double oscillator, a discussion on the analogic situation of the quantum mechanics double oscillator is required. The similarity between these objects will be hopefully as obvious to the reader as it was for the author. Although considerable amount of literature has been written on the subject, it is still useful to recall the essential aspects. In addition the treatment of the quantum mechanics double oscillator is particularly motivated

[1] In the third chapter of this paper we will present such an approach dealing with the phase transition in 3+1 dimensional $\lambda\phi^4$ field theory

when associated with particular phenomena. A concise treatment of the quantum double oscillator following this rationale is done by Merzbacher in one of the most coherent quantum mechanics books *([1]),* despite its considerable age.

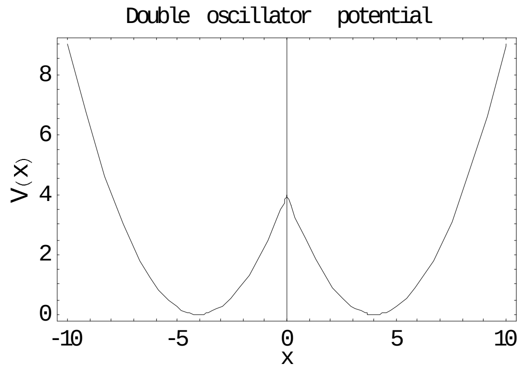
By studying the double oscillator we analyze a more complicated potential, essentially pieced together from two harmonic oscillators. The edge of the discussion is to consider the question of boundary conditions whose effects are observable from the discontinuities arising in the shape of this composite potential.

To start with, most of the immediate applications of the double oscillator quantum theory come from molecular physics. Merzbacher chooses the example of the motion in the neighborhood of a stable equilibrium configuration, which can be approximated by a harmonic potential. Although one-dimensional models are *per se* of limited utility here, important qualitative features can still be exhibited with such a linear model. We consider two masses μ_1 and μ_2 and constrain them to move in a straight line, connected among each other by a spring whose force constant is k and the length at equilibrium is a. If x_1 and x_2 are the coordinates of two designated mass points and p_1, p_2 their respective momenta, we know from classical mechanics that the non-relativistic two-body problem can be separated into the trivial motion of the center of mass and an equivalent one-body motion, executed by a particle of mass $\mu=\frac{\mu_1\mu_2}{\mu_1+\mu_2}$ having a coordinate $x=x_1-x_2$ about a fixed center under the action of the elastic force *[1]* . We will limit ourselves to the relative motion of the reduced mass μ.

The wave equation generated by the equivalent 1-body problem described above has the following form:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2\mu}\frac{\partial^2\psi(x,t)}{\partial x^2}+\frac{1}{2}k(|x|-a)^2\,\psi(x,t) \quad (1)$$

We see immediately that if a=0 , equation (1) reduces to the wave equation for the simple linear harmonic oscillator. When $a\neq 0$ we have “almost” the equation of a harmonic oscillator whose equilibrium position has been shifted by an amount a, but not quite that. We have an absolute value; the potential energy in this case is $V=\frac{1}{2}k(|x|-a)^2$. Such a potential can be easily drawn with a software tool such as Mathematica 4.0. If we give the parameters particular values, we get the graphic below (set k=0.5 and a=4).



From the figure we notice that we deal with two parabolic potentials corresponding (recall the background of this discussion) to the situation where the first particle is to the right of the second particle (id est , x>0), respectively when the particles are in reverse order (id est x<0). The two parabolas are joined at x=0 where the common potential value is $V(0)=V_0=\frac{1}{2}ka^2$. If we take a moment to discuss an already famed difference between the classical and the quantum world, we can make the following observation: classically, if the energy level $E<V_0$, we can actually assume that only one of these potential wells is present, as no penetration of barrier is possible; in quantum mechanics, even if $E<V_0$, the wave function may have a finite value at x=0, which measures the probability that two particles are found in the same place (in other words the barrier can be penetrated). This is nothing else but a specific instance of the EPR paradox.

Before explicitly solving the equation of motion in this case, we assert some obvious observations as a sequel of the remarks stated above. A symmetrical double well like the one we deal with in here will have for energies that are way below the height of the barrier a corresponding pattern of “very nearly degenerate pairs of states” . The ground state with no nodes should definitely have a harmonic oscillator ground state function ω in each well. Each of these ground state function should go practically to 0 in the middle region where they meet smoothly. If we think ahead of the first excited state with one node we should have almost the same energy as in the ground state. The difference is that each wave goes to 0 in the middle region, but actually crossing the axis, while meeting smoothly in the middle and thus producing the node[2].

The explicit Schrodinger equation corresponding to the wave equation (1) above (not depending on time) reads :

[2] It should be clear that in this first excited state the wave function will have a harmonic oscillator ground state function ψ in one side and $-\psi$ in the other side, so that they can cross the axis.

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2\psi(x)}{\partial x^2}+\frac{1}{2}k(|x|-a)^2\,\psi(x)=E\psi(x) \quad (2)$$

For |x|>>a, equation (2) approaches the Schrodinger equation for the simple harmonic oscillator

$-\frac{\hbar^2}{2\mu}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{2}\mu\omega^2 x^2\psi=E\psi$; therefore the physically acceptable eigenfunctions must be required to vanish as $|x|\to\infty$.

Note that as a is varied from 0 to ∞, the potential changes from the limit of a single harmonic oscillator well to the other limit of two separate oscillator wells divided by an infinitely high and broad potential wall (an observation also made above, when trying to “guess” the pattern of the solution). In the simpler case we have non-degenerate energy eigenvalues *[1]:*

$$E_n=\hbar\omega(n+\frac{1}{2})=\hbar\sqrt{\frac{k}{\mu}}(n+\frac{1}{2})\text{ , where n=0,1,2...}$$

In the other extreme case, when we think of two separate oscillator wells the energy values shall be exactly like these ones above only that each of them will be doubly degenerate since the system can occupy an eigenstate of either one of the two wells. As the parameter *a* is varied, the energies and eigenfunctions will change continuously between the 2 limiting cases. This is the *adiabatic* change of the system *[1]*. Nonetheless, as the potential is being distorted, certain features of the eigenfunctions remain unaltered. An example of this sort of *adiabatic invariants* is the number of nodes of the eigenfunctions. Indeed, given that one eigenfunction has n nodes, it cannot change this number in its transition from the potential in the extreme lower case to the extreme higher case. The proof of this assertion is immediate and can be found in a very detailed description in *[1]*, page 68. Extremely interesting is the consequence of this remark, namely that being an adiabatic invariant, the number of nodes characterizes the eigenfunctions of the double oscillator for any value of *a*. A rigorous solution for providing the eigenvalues and the eigenfunctions is introduced by splitting the cases for positive and negative coordinates *[1]*. Thus, for positive x we introduce

$$z=\left(\frac{4\mu k}{\hbar^2}\right)(x-a)=\left(\frac{2\mu\omega}{\hbar}\right)^{\frac{1}{2}}(x-a) \quad \text{and} \quad E=\hbar\omega(\nu+\frac{1}{2})$$

For negative x we have almost the same equation, with a small difference in the substitution relation:

$$z'=\left(\frac{4\mu k}{\hbar^2}\right)(x+a)=\left(\frac{2\mu\omega}{\hbar}\right)^{\frac{1}{2}}(x+a)$$

By differentiating in both cases we obtain the following equations:

$\frac{\partial^2\varphi}{\partial z^2} - +(\nu + \frac{1}{2} - \frac{z^2}{4})\varphi = 0$, respectively $\frac{\partial^2\varphi}{\partial z'^2} - +(\nu + \frac{1}{2} - \frac{z'^2}{4})\varphi = 0$. It is plain as day that for a=0 the 2 equations above become identical and we deal again with the harmonic oscillator.

The originality of the approach in Merzbacher rests in the method employed for solving the differential equation above. Certainly the incipient idea would be to proceed with a detailed power series treatment. Merzbacher uses instead a *parabolic cylinder function* in order to find a particular solution. This function is defined as:

$$D_\nu(z) = 2^{\nu/2} e^{-(z^2/4)} [\frac{\Gamma(\frac{1}{2})}{\Gamma[(1-\nu)/2]} {}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; \frac{z^2}{2}\right) + \frac{z}{\sqrt{2}} \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{\nu}{2})} {}_1F_1(\frac{1-\nu}{2}; \frac{3}{2}; \frac{z^2}{2})],$$

where ${}_1F_1$ is the confluent hypergeometric function. The function is expandable in power series as follows: ${}_1F_1$ (a;b;z)=1+$\frac{a}{b}\frac{z}{1!} + \frac{a(a+1)z^2}{b(b+1)2!} + ... = \sum_{k=0}^{\infty} \frac{a_k}{b_k}\frac{z_k}{k!}$

As our purpose in this section is more to outline the reasoning and the originality of the approach rather than describing the technical subtleties, we will keep to that, leaving the unsatisfied reader with the possibility of consulting himself the further reference for this section. The underlying reasoning is constructed as follows: if $D_\nu(z)$ is a solution of the discussed differential equation above than immediately $D_\nu(-z)$ is a solution of the same equation and moreover these solutions are linearly independent unless ν is a nonnegative integer. It follows that a double oscillator eigenfunction must be proportional to $D_\nu(z)$ for positive values of x and proportional to $D_\nu(-z)$ for negative values. It remains to join these solutions at x=0, this being the point where the two parabolic potentials meet with a discontinuous slope. We investigate therefore the singularity of this point. Since the Schrodinger equation is a second-order differential equation, φ and its first derivative must be continuos. In other words φ must belong to the C^2 class of functions with continuous first derivatives. By extension, if $x = x_0$ is indeed a singularity point, we can integrate the Schrodinger equation from $x = x_0 - \varepsilon$ to $x = x_0 + \varepsilon$. Then

$$\psi'(x_0 + \varepsilon) - \psi'(x_0 - \varepsilon) = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{2\mu}{\hbar^2}[V(x) - E]\psi(x)dx$$

The immediate contention is that as long as V(x) is finite the equation above implies that ψ' is continuous across the singularity. It follows that ψ should also be continuous. We can further assume that the eigenfunctions have definite parity, even or odd. If an even function of x has a continuous slope at x=0, as the joining condition requires, that slope must be 0. On the other hand from basic analysis it follows that if an odd function of x is continuous at the origin, it must vanish there. Thus, by matching

ψ and ψ' at x=0 leads at the following transcendental equations for v:

$$D_v'(-\sqrt{\frac{2\mu\omega}{\hbar}}a) = 0, \text{ if } \psi \text{ is even and}$$

$$D_v(-\sqrt{\frac{2\mu\omega}{\hbar}}a) = 0 \text{ if } \psi \text{ is odd.}$$

In general it is difficult to calculate the roots v of the equations above. Explicit formulas can be obtained if $V_0 >> E$ for instance. The unnormalized eigenfunctions can be still written in function of these above equations. Thus,

$$\psi(x) = D_v(\sqrt{\frac{2\mu\omega}{\hbar}}(x-a)) \text{ for x>=0 and } \psi(x) = \pm D_v(-\sqrt{\frac{2\mu\omega}{\hbar}}(x+a)) \text{ for x<=0.}$$

All in all, we have followed the Merzbacher's reasoning in finding the eigenvalues and eigenvectors of the quantum mechanics double oscillator and attained a preliminary goal. In what follows we will shift from the quantum mechanics to the quantum field theory world, discussing the physics of the self-interacting scalar fields.

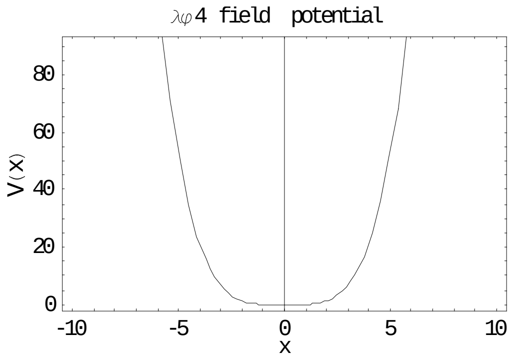
3. The physics of the self-interacting scalar fields

We depart in our analysis from the by now universally agreed fact that all particles in the Standard Model acquire their masses from a non-vanishing expectation value $<\phi>$ of a self-interacting scalar field and we follow the reasoning of Consoli and Stevenson in this respect*[2]*. The idea is considered relatively simple and has already had a long-history behind; nonetheless the nature of the phase transition in the $\lambda\phi^4$ scalar field theory remains a hard task and must be given careful consideration.

Looking through the perspective of the classical observer we only need to consider a potential of the following form:

$$V_{cl}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

If we vary the m^2 parameter, we can notice that the phase transition will actually occur at $m^2 = 0$. Indeed, a graph of that potential for $m^2=0$ and $\lambda = 1$, with ϕ being varied from –10 to 10, for instance, would look as follows .



Obviously the assumption above can only be satisfactory in the realm of the classical physics. In the quantum theory the question is more subtle. In particular, even if we know that the symmetric vacuum is locally stable if $m^2>0$, we cannot be sure that this symmetric vacuum is necessarily globally stable. In other words we need to ask ourselves whether the phase transition could actually be of a first order and hence occurring at some small, but positive m^2 *[2]*.

The standard approximation methods for the quantum effective potential are not appropriate in this situation *[2]*. It has been suggested that the Gaussian method provides a clue, producing a result in agreement with the one-loop effective potential in 3+1 dimensions *[2]*. The basic idea adopted by several authors is based on the "triviality" of the continuum limit of $\lambda\phi^4$. The immediate implication of this presumption would be that the effective quantum potential should be physically indistinguishable from the classical potential plus some "zero-point-energy" contribution of free field form arising from fluctuations (which would also lead to the fact that all approximations using this assumption are in the end equivalent). In a mathematical form the trivial potential would be:

$$V_{triv}(\phi)=V_{cl}(\phi)+\frac{1}{V}\sum_k \frac{1}{2}\sqrt{k^2+M^2(\phi)}\ ,$$

where $M(\phi)$ denotes the mass of the shifted field $h(x)=\phi(x)-\phi$, in the presence of a background field ϕ. After mass renormalization and subtraction of a constant term *[2]*, $V_{triv}(\phi)$ consists of ϕ^2, ϕ^4, and $\phi^4 \ln \phi^2$ terms. The very point of this contention is that any detectable difference in this model would imply interactions of the h(x) field. However, as we assumed that the theory is 'trivial', we are not suppose to obtain such interactions. Then it must be that there is an infinite class of "triviality-compatible" approximations, all yielding the same result. Such approximations can be arbitrarily complex provided that they have a variational structure, with the shifted field h(x)= ϕ(x)- ϕ having a propagator determined by solving a non-perturbative gap-equation. If the approximation is "trivially compatible" *[2]*, this propagator reduces to a free-field propagator in the infinite-cutoff limit. In that limit all differences among these various approximations can be absorbed into a redefinition of the parameter λ (and this was our aim when working with the assumption of 'triviality'), which makes no further difference when the effective potential is expressed in terms of physical renormalized quantities.

If we now return to the introduction of this topic and suppose that spontaneous symmetry breaking does indeed coexist with a physical mass $m^2 \geq 0$ for the excitations of the symmetric phase, those excitations would actually be real particles. These "phions" will play the main role in the reformulation of our initial question: how is it possible for the broken-symmetry vacuum, a condensate with a non-zero density of phions, to have a lower energy density than the 'empty' state with no phions at all? The solution rests after Consoli and Stevenson in the fact that the phion-phion (or $\lambda\phi^4$ interaction) is not always repulsive, but there is also an induced interaction that is attractive. Moreover it is secured that as m→0 the attraction becomes so long range ($-1/r^3$), that it generates an infrared-divergent scattering length. Following this rationale, the long-range attraction makes it energetically favorable for the condensate to form spontaneously. And this constitutes nothing else but a physical mechanism for spontaneous symmetry breaking.

Consoli and Stevenson do not stop here in their paper. It is contended furthermore that "even an infinitesimal two-body interaction can induce a macroscopic range of the ground state if the vacuum contains an infinite density of condensed phions". This is consistent with the condensate density being infinite in physical length unit *[3]*. Further using units with $\hbar = c = 1$ and the single-component $\lambda\phi^4$ theory with a discrete reflection symmetry, $\phi \rightarrow -\phi$, which is consistent with our approximations, the inter-particle potential between the phions is discussed. An estimate of the energy density of a phion condensate is subsequently achieved in an intuitive way. The research paper concludes with a section on the phases transition resulted from the field-theoretic effective potential, including a discussion on how this effective potential can be written in a finite form in terms of the renormalized field. We will re-assert herein the results concerning the inter-particle potential and the implications on the phase transition.

Consoli and Stevenson notice that the inter-particle potential is essentially given by the sum of a so-called "repulsive core", $\delta^{(3)}(r)$, and an "attractive part ", $\frac{-1}{r^3}$, that is eventually cut off exponentially at distances grater than $\frac{1}{2m}$. As an exact expression of this long-range attractive potential the following has been found: $V_{long-range}(r) = -\frac{\lambda^2}{256\pi^3 E^2}\frac{1}{r^3}$.

A very important result of this long-range interaction resides in the expression of the ground state energy density for a large number of phions (N) in a large box of volume V with a fixed density n=N/V. The way this ground energy density is reduced follows from the assumption of considering n low enough so that the rest-masses Nm and the 2-body interaction energies form alone the total energy in the ground state[3]. In other words the contention above can be reformulated as $E_{tot} = Nm + \frac{1}{2}N^2\bar{u}$, where $\bar{u}$ is the average potential energy between a pair of phions: $\bar{u} \approx \frac{1}{V}\int d^3 r V(r)$. Provided that m is very small, we might have that even though the empty state is locally stable, it might decay by spontaneously generating particles so as to fill the box with a dilute condensate of a non-zero density. We find extremely relevant the translation of the particle density into field theory ($n = \frac{1}{2}m\phi^2$). The energy density as a function of n becomes then the field theoretic effective potential[4]. The result is embedded in the following expression:

$$V_{eff}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda\phi^4}{32} - \frac{\lambda^2\phi^4}{256\eta^2}\ln\frac{r_{\max}(\phi)}{r_0}.$$

Consoli and Stevenson actually prove the expression of the efficient potential (derived above on an intuitive basis) by using a relativistic version of the original Lee-Huang-Yang analysis of the Bose-Einstein condensation of a non-ideal gas *[2]*. We will not insist on this rather technical approach and will further focus on the discussion of the phase transition. $V_{eff}(\phi)$ is found to have an important qualitative difference when compared to the classical potential. This is extremely interesting as the present paper focuses on classical solutions to the double oscillator field theory and prepares the ground for further comparison between the quantum mechanics and quantum field theories on the one hand and the classical approach, on the other hand. Concretely, the classical potential has a double-well form only for negative m^2 values and has a phase transition of second order at the value m^2=0.

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[3] Although not every researcher would agree, the basis of the assumption here is that the gas of phions is dilute and therefore effects from three-body or multi-body interactions will be negligible

[4] The energy density as a function of n can be found by setting E=m since almost all phions have k=0. Hence Consoli and Stevenson obtain the result: $\varepsilon = nm + \frac{\lambda n^2}{8m^2} - \frac{\lambda^2 n^2}{64\pi^2 m^2}\int\frac{dr}{r}$.

With $V_{eff}(\phi)$ on the other hand, the phase transition occurs at $m^2 = m_c^2$, where

$$m_c^2 = \frac{\lambda^2}{128\pi^2}\frac{v_0^2}{\sqrt{e}} \geq m^2 \ .$$

The resulting form of the effective potential and the value at which the symmetry is broken are imminent in a "trivial" theory such as the one employed by Consoli and Stevenson. However, despite this triviality, a rich hierarchy of length scales is found to emerge. This hierarchy is perfectly summarized in *[2]*.

An interesting discussion on the renormalized form of the effective potential is conducted in the last section of the Consoli-Stevenson paper. Using a renormalized field $\phi_R = Z_\phi^{-1/2}\phi_B$ (for theoretical background one can consult *[10]*) , where Z_ϕ is a re-scaling factor, the effective potential can be written in a manifestly finite form. A finite parameter ς is defined ahead in this respect: $\varsigma = \dfrac{M_h^2}{8\pi^2 v_R^2}$.

Imposing the necessary boundary conditions we find as final form of V_{eff} the following:

$$V_{eff}(\phi_R) = \pi^2\varsigma(\varsigma-1)\phi_R^2(2v_R^2 - \phi_R^2) + \pi^2\varsigma^2\phi_R^4(\ln\frac{\phi_R^2}{v_R^2} - \frac{1}{2})$$

We can see immediately that in the extreme case $\varsigma \to 0$ we actually deal with the classical potential result. It is computed that the symmetry-breaking phase transition occurs at $\varsigma = 2$ (which corresponds to the value $m^2 = m_c^2$). What was achieved by renormalization is in fact an intrinsic parameterization of the effective potential by the two independent quantities ς and v_R^2 (the vacuum expectation value). They replace the two bare parameters m^2 and λ of the original Hamiltonian. As a final observation, it is interesting (also for the purpose of this paper) to discuss the range 1> ς >0 which corresponds to negative values of m^2. This is the range of the so-called tachionic phions. A graph for the case m^2 <0 has been introduced in the introduction chapter when investigating in a first-analysis perspective the correlation between the $\lambda\phi^4$ theory and the double oscillator field theory.

In this chapter we have thus analyzed the self-interacting field theory around its symmetry breaking values. However we did not question ourselves yet how sensitive would this phase-transition be in the presence of sources of the scalar field. A traditional concrete example in this sense is the much discussed electroweak model where these sources are materialized in quarks or leptons. In the next section we introduce a similar self-standing model, the double oscillator field theory model. It was already acknowledged in the previous paragraphs that phase-transitions usually lead to discontinuities.

We will investigate whether an intermediate stage in this phase-transitions and subsequently, a well behaved model to account for these discontinuities, can be constituted by classical scalar field sources, namely solutions to the double oscillator field model. In this paper we investigate simple as well as composite solutions in the realm of the field theory double oscillator.

4. Simple solutions to the double oscillator field theory

After an in-depth discussion of the quantum double oscillator and the $\lambda\phi^4$ field theory, pointing out necessary similarities and differences, we shall aim at finding exact solutions to the double oscillator field theory. We first investigate the existence of the simple solutions, that is regular charges (the almost trivial case) and the regular bubble solutions, in other words the solutions corresponding to the phase transitions in the $\lambda\phi^4$ theory discussed above.

Before getting more concrete, we ought to clarify the pragmatics of our research. It should be by now clear what the use of the classical solutions can be in this situation. Firstly, the classical solutions (with special emphasis on the bubble solutions) are related to N-boson production amplitudes; to state it otherwise, we are testing herein whether we eventually deal with “bubbles of bosons”, that is whether we deal with bound-states of a huge number of bosons, condensed around a charge *[8]*. What exactly are these bubbles? Probably a perfect definition cannot be found, nonetheless they can be described as quantized droplets of a different vacuum phase, which at the same time are non-perturbative resonant states of the field investigated *[7], [8]*. Secondly, these classical solutions play a very important role as intermediate products in the phase-transitions in general and, as a definite application, in the early universe. The cosmological background represents a challenging research ground in this sense *[6]* . In the light of all these reasons, we find ourselves very motivated in assessing the existence of this sort of solutions. As an additional observation, the fact that non-perturbative means are employed is again a potential improvement to usual approaches.

Let us first introduce the following equation of motion (equation that will play the guiding role in our further calculations):

$Eqm(\phi)=\partial_t^2\phi-\nabla^2\phi+m^2\phi-m^2aSign(\phi)=J(r)$ (*),

where *J(r)* is a source of the form $Q_0\delta(\vec{r})$ that enables this motion. In what follows we discuss in consecutive subsections the regular charges, respectively the regular bubble solutions to this equation. We have some common provisions that will apply to both subsections. Firstly, we use a spherical coordinate system, as intuitively it is more than clear that the dependence shall remain on *r* alone for the simplest cases (applications in *[1]*, *[12]*). Secondly, in order to simplify the understanding and the consequent computations, we denote solutions to the equation of motion above by[5] {{ sign of charge; sign of vacuum}$_{\text{in}}$ and {{sign of charge, sign of vacuum}}$_{\text{out}}$. Then we can immediately identify which cases will correspond to possible regular charges and which correspond to regular bubble solutions (same sign of charge and vacuum means a single charge solution).

As far as regular charges are concerned we will have the cases:

$\Lambda_+ = \{(+,+),(+,+)\}$, respectively $\Lambda_- = \{(-,-),(-,-)\}$

Logically, it follows that the regular bubbles are:

$\Xi_+ = \{(+,+),(+,-)\}$

$\Xi_- = \{(-,-),(-,+)\}$

$\Xi^+ = \{(+,-),(+,+)\}$

$\Xi^- = \{(-,+),(-,-)\}$

It should be clear from the reasoning above that these are the only possible configurations having a single bubble wall (that is, they contribute as "simple" solutions to the total solution space)

Thirdly, we particularize the problem in that we search for solutions generated by sources of the particular form $\phi = \eta(x) \pm a$. We contend that generality loss does not necessarily happen as result of imposing such a restriction, as the conclusions drawn in the end are again subject to generalization.

4.1. Regular charge solutions

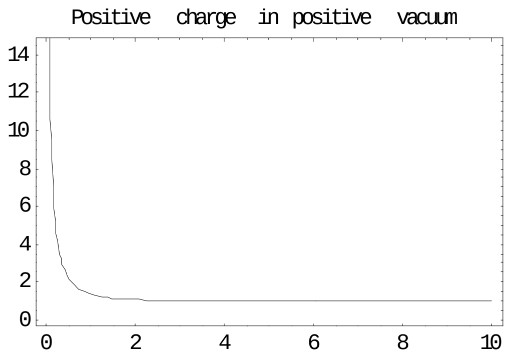
In order to separate the two charge solutions we introduce a positive test charge Q_0. Subsequently our regular charges will be of the form:

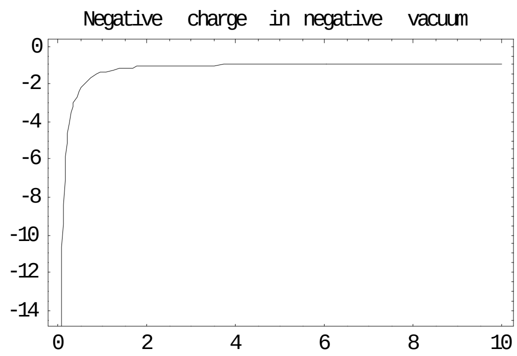
$\Lambda_+(r) = Q_0 \dfrac{e^{-mr}}{mr} + a$, respectively $\Lambda_-(r) = Q_0 \dfrac{e^{-mr}}{mr} - a$ (the Yukawa expression is in agreement with the restriction on the source $\phi = \eta(x) \pm a$ imposed above, being also a standard form used in this sort of computations)

All that remained to do is substitute these solutions in the equation of motion exposed above and check that we are indeed dealing with "regular charges", that is charges in a vacuum of equal

[5] The notation herein has been inherited from the previous notes on the subject, of Dr. Frank Witte

signature. In order to be able to assess the correctness of these solutions, we will use a qualitative interpretation, plotting them:





We notice that the representation is in conformity with our assumption, hence we can conclude that the ground state with a certain sign supports Yukawa charges with an equal sign. It needs to be noted however that herein we do not take into discussion time-dependent solutions, thus making a trivial assumption of the vacuum condensate being time independent. Nonetheless we can safely contend that regular charges of the given form are indeed solutions to the double oscillator field theory.

4.2. Regular 'bubble' solutions

If the regular charges solutions were approaching triviality, in the case of the regular bubbles we are faced with a considerable heavier task. We start using the same intuitive reasoning trying to "guess" the form of these second-class solutions. Getting back to the background of our research, we recall that we talk about symmetry breaking in a field theory. As it was brought forward in preceding paragraphs, this symmetry breaking arises possibly in the form of discontinuities. If we match a charge into a vacuum of a different sign, there will definitely be a discontinuous jump in the second space derivative of the field. However, not all these cases are necessarily consistent with the model and that requires careful investigation.

Essentially, the bubble solutions could exist where the potential of the double field oscillator has two minima. Locality does not play a role here, the minima being absolute in the case of the double field oscillator[6].

[6] This is taken in contrast with the self-interacting field theory, where Consoli and Stevenson clearly contend a questionable single minimum, at least in the first place. See the corresponding chapter above or refer to the paper by the mentioned authors (*[2])*

We use again a test charge Q as a strictly positive constant and we define the possible types of “bubbles” in function of this test charge. We use the following framework solutions (which will constitute substantial solutions once the vacuum value has been accounted for):

$\Phi_{in}(r) = \frac{Q}{mr}(c_1 e^{-mr} + c_2 e^{mr})$, while

$\Phi_{out}(r) = \frac{Q}{mr} e^{-mr}$

where the two equations stand for the inside, respectively for the outside solutions within the field.
It is clear that the bubble wall is located where the effective potential becomes 0 and where the vacuum is behaving “unnaturally”. In this chapter we solely consider single charges in our system, thus the bubbles will depend only on the radius r generated by these point charges.

In function of the inside/outside solutions above, we can define the bubbles. It’s simply logical that we can only have four kinds of them. They bare the same framework form in terms of exterior and interior solutions. Namely in both cases the Yukawa potential *[3], [12],* $\frac{e^{-mr}}{mr}$ is decisive (as one can readily notice in the framework solutions above), the vacuum value a making the difference (as being added or subtracted, respectively). The possible cases are the following:

$\Xi_+(r) = \Phi_{in}(r) + a$, if r<R and $\Xi_+(r) = \Phi_{out}(r) - a$, if r>R

$\Xi_-(r) = -\Phi_{in}(r) - a$, if r<R and $\Xi_-(r) = -\Phi_{out}(r) + a$, if r>R

$\Xi^+(r) = \Phi_{in}(r) - a$, if r<R and $\Xi^+(r) = \Phi_{out}(r) + a$, if r>R

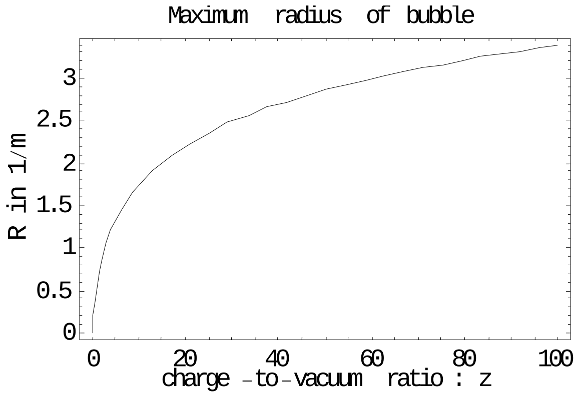
$\Xi^-(r) = -\Phi_{in}(r) + a$, if r<R and $\Xi^-(r) = -\Phi_{out}(r) - a$, if r>R,

where R is the radius of the bubble.

Let us take a closer look to each of these solutions.

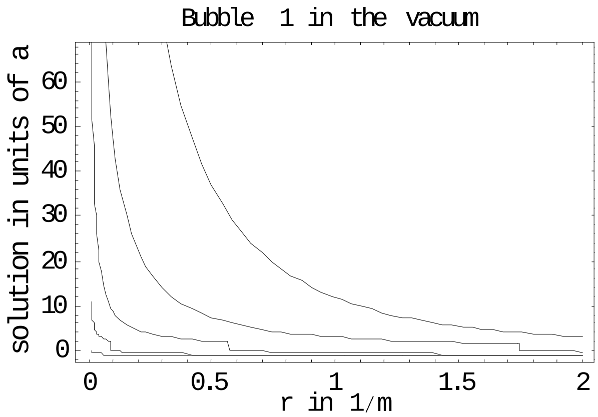
For the first type of bubble solutions, $\Xi_+(r)$, the outcome is the following.
Checking the values for the coefficients c_1 and c_2 and consequently computing the field, we find that R is restricted as a function of the charge to vacuum parameter $z = \frac{Q}{a}$. We also need to have, following the conditions for the interior of the bubble, that $\frac{e^{-mr}Q}{mr} - a \le 0$, which puts an upper limit on the radius. Subsequently, if the value for r drops below this limit the outer field will no longer be a

solution to the equations of motion. The maximum radius is thus found as a limit solution to the equation above. A qualitative interpretation of the maximum radius in function of the charge-to-vacuum ration has been plotted below:



We see in a clear way the dependence between the maximum radius of the bubble and the charge-to-vacuum ratio. By analogy to the previous chapter on self-interacting scalar fields, this charge-to-vacuum ratio obviously plays the role of v_R^2, the so-called vacuum expectation value. An increased ratio would increase the radius in inverse mass units.

As the properties of this bubble strictly depend on the charge/vacuum parameter, we are definitely interested in its qualitative behavior for certain values of this parameter. A plot for a few values is presented below:

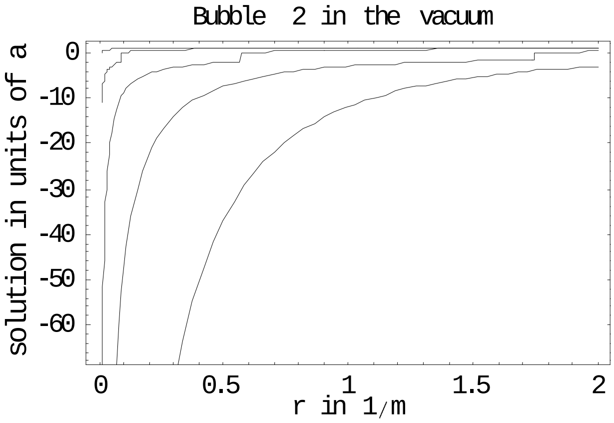


Hereinabove the considered values of z were 10^{-2}, 10^{-1}, 10^{0}, 10^{1}, 10^{2}.

The graph is speaking for itself: the bubble here is represented by a positive charge in a negative vacuum. Or this was the initial contention when we departed in analyzing this type of bubbles.

In the light of the foregoing, we note that the existence of the bubble is not questionable (the bubble behaves as a positive charge in a negative vacuum) and moreover we are reminded again that the bubble wall is always located at the null point of the field.

Further we need to check the second possible type of bubble solutions, $\Xi_{-}(r)$. Reasoning in a similar way as in the paragraphs above, we get again a restriction on the range of values for R, the parameter being again the charge-to-vacuum one. On the other hand, checking for the exterior boundary, we get the same condition as in the case of the first type of bubble solution. We can thus directly investigate the qualitative behavior (imposing the same charge-to-vacuum parameter values as in the preceding section):



From the graph above it is more than clear that the second type of bubble solution is representative for bubbles behaving as negative charges in a positive vacuum, in other words the opposite of what we obtained in the case of the first type of bubbles (they are simply symmetric states in terms of the framework solutions)

We arrived at the case of the bubble solutions $\Xi^{+}(r)$, with the bubble wall appearing in the left side of the conventional notation. We follow the same steps as in testing the preceding cases. R will still depend on the charge- to- vacuum parameter but this time in a different manner than we had before. A straightforward computation indicates that R is actually a strictly monotonous function of the charge-to-vacuum parameter[7]. It is worth noting that checking the existence of the bubble from the outside we obtain an identity, hence the bubble could in principle exist in the field, but nonetheless, as previously proved, it would not be sustained from the interior. And for a bubble to exist the combination between the exterior and interior condition has to be fulfilled. Thus bubbles of this type do not exist as proper solutions to the considered equation of motion.

[7] The mere thing we are required to do here is to study the zero's of the function: -

$1+\dfrac{e^{-r-R}(e^{2R}(-1+R)+e^{2r}(1+R)+e^{R}z)}{r}$. We obtain that this is monotonically increasing and has no zeros.

We could already reason by means of symmetry in the case of the last type of bubble. Evaluating the conditions, we find that the same interior equation does not have any solutions in that interval. However, from an “outdoor” perspective, the field would allow this bubble to exist as the condition for the exterior of the bubble is always satisfied.

We observe that in the last two cases no match of the interior to the exterior solutions is needed, since for all values, the interior condition cannot be fulfilled. Thus this step is superfluous.

In conclusion we contend that as far as regular bubbles are concerned, id est charges in vacuums of different signs (with single bubble wall), we can have two possible cases, namely

$\Xi_+ = \{(+,+),(+,-)\}$ and

$\Xi_- = \{(-,-),(-,+)\}$.

Therefore the set of solutions to the double oscillator field theory comprises so far regular charges and single wall bubbles, where the latter can exist only in the first two instances.

So far we have ignored the possibility of a second charge in our system ; we have discussed only the simple solutions to the double oscillator field theory where the upshot could be simply regular charges (or charges in vacuum of the same signature) or regular bubbles (charges in vacuum of different signature). These regular bubbles were all spherical and had symmetry properties, as they were condensed around a single point charge. What happens however when a second charge is added? How does the physics of the field modify? What will be the location of the bubble wall in this case? We try to come with meaningful answers to these questions in the next chapter.

5. Composite solutions to the double oscillator field theory

In so far we have treated only regular solutions to the double field oscillator field theory, namely regular charges and regular bubbles. As already introduced in the preceding chapter, should we consider a second charge, the spherical symmetry of the bubbles will intuitively be disturbed. We will thus have to incorporate in our total set of solutions so called composite solutions. We can think of at least three category of phenomena interesting to be studied in this respect: interactions between bubbles and regular charges, interactions between bubbles as such and dipole solutions. The ‘moment generating function’ of all these phenomena will be the structure formed by the 2 charges considered, with focus on their separation distance. In what follows we will study the extremum cases (charges superposed or very close, respectively charges situated extremely far away from each other) and then we will get to the more difficult case of studying the character of the solution in function of their “separation” distance parameter.

Given that the double oscillator theory is in essence a linear theory (linearity is preserved when transiting from quantum mechanics to quantum field theory), the potentials might be *in principle* superposed. We emphasize “in principle” as we need to be extra careful where the total sum of the potential becomes 0 and where subsequently we need to fit the interior to the exterior solution[8]. It is very likely that in the realm of the composite solutions, the interior solution will depend on more than just r, hence a spherical coordinate system is not appropriate when studying the behavior of the double oscillator field theory herein. Instead, we will try to solve the system in a cylindrical fitting. We expect rotational symmetry around the charges axis. In what follows we investigate this types of solutions focusing on the behavior of the exterior solution, as we are interested in the existence of these bubbles first of all from the field perspective.

5.1. Extremum cases

The equations that we will use for the exterior, respectively the interior solution, are of course based on the corresponding equations for the spherical bubbles case, with the exception that this time the bubble will also depend on the *z* parameter (as we expect rotational symmetry around the *z* axis). We will herein consider the exterior solution, in order to inspect if the bubble could at all exist :

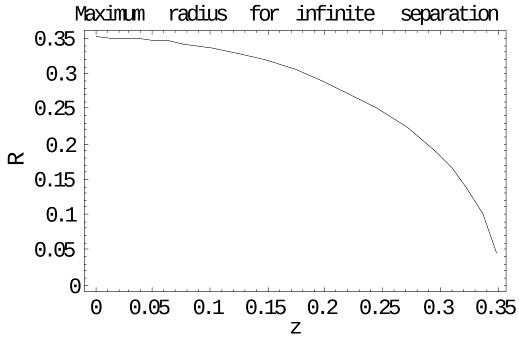
$$\Phi_{out}(r,z) = -a + Q_1 \frac{e^{-m\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} + Q_2 \frac{e^{-m\sqrt{r^2+(z-k)^2}}}{\sqrt{r^2+(z-k)^2}} ,$$

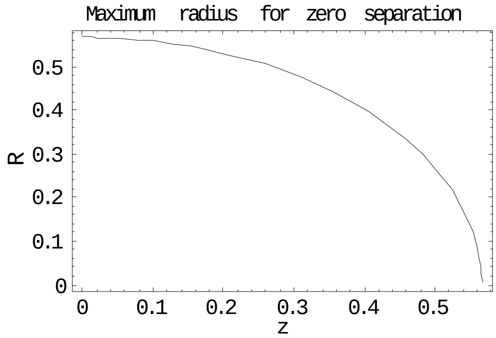
where *k* is the separation parameter between the two charges (we assume of course that $k \ll R$).

Before analyzing the exterior potential in more detail, we make the remark that the interior solution will also be constructed on the Yukawa potential, hence it is build on a similar framework as the interior solution for the regular charges; finding non-perturbatively this solution is more than causing trouble, hence we will assume for the moment that the existence of this solution is not questionable and thus we can investigate the overall solution by pinpointing to the exterior one.

We start by looking closer to the cases where the separation parameter between the existing and the incoming charge are 0, respectively ∞. This cases are easy to treat and the results will reduce to simple solutions of the double oscillator field theory. We notice that for k=∞ the second term completely vanishes and we have to deal with a single charge in a bubble, or in other words with an upshot described as a bubble condensed around this charge. This type of solution was already analyzed in the previous section, the solution here being thus of a similar pattern. Taking k=0 we notice that the two charges simply add up (superposition of potentials). Hence we have the same case as previously, with the observation that if the charges are opposite, in an obvious way they will cancel each other. We will henceforth investigate these two cases altogether.

If we are to plot the evolution of the radius in function of *z* for the cases where the separation parameter is ∞, respectively 0, we will be surprised to notice that they behave almost identically (ignoring the scale difference, of course). And this is after all a simple logical consequence of the fact that superposed charges or charges at an infinity distance will "produce" a bubble with similar features, namely a bubble condensed around the charge taken as reference.





We can see the similarities in the plots above, where the parameters were all set equal to unity, for simplicity[9]. What we also notice is that the radius has exactly the same pattern as we discovered with regard to the regular bubbles; this is again simply following from the fact that extreme cases boil down to simple solutions. Let us look for a moment at the expression of the radius in function of the *z* coordinate for the cases k=0 and respectively k=∞. We have

$$r \to \frac{\sqrt{-m^2 z^2 + \mathrm{ProductLog}\left[\frac{m(Q1+Q2)}{a}\right]^2}}{m}$$, for k=0 and

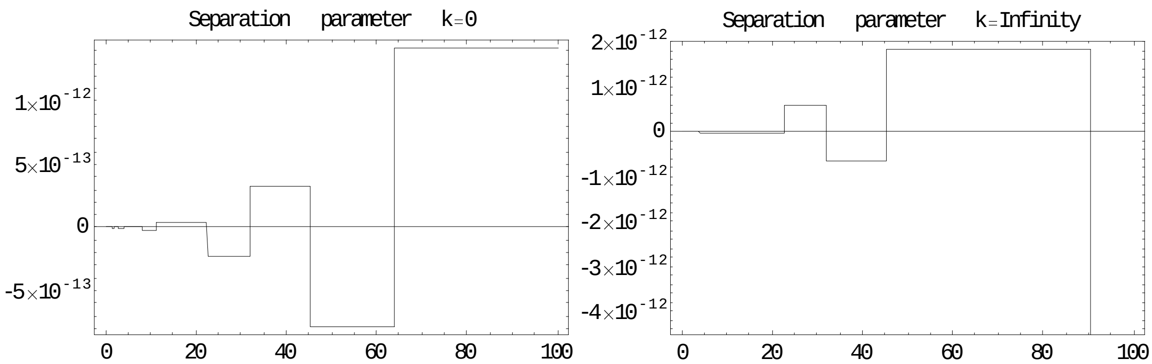
$$r \to \frac{\sqrt{-m^2 z^2 + \mathrm{ProductLog}\left[\frac{mQ1}{a}\right]^2}}{m}$$, for k=∞.

In both expressions above, *ProductLog(x)* gives the principal solution for *u* in the equation $x = ue^u$.

[8] We have insisted already in matching the solutions in the preceding sections, when treating the simple solutions to the double oscillator field theory. Hence no supplementary explanation is given here.
[9] In general of course R will be represented in units 1/m, but as we set here m=1 we can ignore them

We clearly observe now the relation between these 2 extreme cases. We also see that by setting all parameters equal to unity (implying that we work with unity charges as well), the effect of the second charge in the case k=0 falls from contributing in a significant manner and hence the almost identity in the 2 plots.

We expect the behavior of the exterior solution to copy more or less the pattern of the one in the regular charges. Of course there will be some noise added because of the additional charge. We leave to the ambitious reader the task of manipulating the equation and find the framework solution (we suggest the use of the Mathematica family software as the computations are otherwise an impossible task). Hereinafter we reproduce the plot of the exterior solution functions for the extreme cases using the same values for the parameters (m=1, Q_1=1, Q_2=1, a=2).

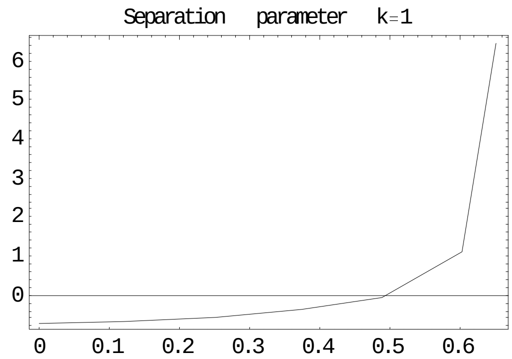


There is without any doubt more to discuss about extreme situations, however a much more interesting and challenging task is to see what happens while we vary the separation parameter between the 2 charges of our system. The next section gives an overview on this aspect.

5.2. Complex interaction cases

To our disappointment solutions to the case of complex interactions were not found with the same precision as before. While non-perturbative methods applied in an analytical framework failed to give any desired results, numerical methods did not perform better, achieving results only for the limiting cases discussed above. In other words, once we start the discussion on the complex cases where the separation parameter is not 0 or ∞, traditional numerical methods such as Newton's or the secant method, fail.

However, given that more than satisfactory results were obtained as far as simple solutions are concerned, we can try to reason on the complex situation using the heuristics provided therein. It is plain as day that the simple solutions can be taken as limits to the complex cases. Expanding on this idea, if we consider close solutions to the limiting cases discussed in 5.1 we can try to use the same estimate for the radius, modifying solely the value of the separation parameter. In this spirit, let us consider the case were the separation parameter would be unity. We plot the exterior solution in what follows; we use the same reference value of the radius as for the limiting case k=0.



We notice the difference from the limiting case. Due to the fact that the choice of the parameters has to be extremely careful in order to get a meaningful plot, we can say that the graph might not reproduce a perfect situation; moreover we have the solution in terms of *z*, hence we cannot compare it directly to the similar solution within the regular bubble realm. Nevertheless, we can 'guess' that the bubble will become an ellipsoid rather then a sphere and with increasing separation parameter will lose more and more of its unity until in the end will become two separate parts, or a pure dipole.

We leave the further investigations of these cases to a sequel of this paper and in what follows we try to present the conclusions to our research, not before discussing the application background and of course, the issue concerning the stability of the bubbles.

6. Discussion and Conclusions.

We have analyzed in this paper the types of classical solutions to the double oscillator field theory. A discussion on the importance of the classes of solutions in the context of the quantum field theory has not been done yet, however. We shall not leave such an important issue uncovered and shall treat in what follows the background of this research.

According to the standard model and to its extensions, symmetry breaking phase transitions are expected to have occurred on a massive scale in the early universe. It is a known fact that the mechanism by which these transitions can happen can be spinodal decomposition *[7]* or the formation of the bubbles of the new phase of the universe with the old one. Or this is the most amazing application of our bubble solutions, largely discussed in this paper. The vacuum stages separated by the bubble wall herein are nothing but the early universe and the actual universe. It is true that the bubble theory is commonly accepted especially as far as the electroweak phase transition is concerned, while the spinodal decomposition is favored otherwise; nevertheless the generating mechanism is interesting to study for all cases. To finish our idea about the formation of the universe, the phase transition bubble will expand and collide with each other until the whole volume is occupied[10], at which time the transition early universe→ actual universe is considered fulfilled.

Another issue that was not brought in for discussion but certainly has its scientific merits is the stability of the bubble solutions to the double oscillator field theory. Configurations of the regular bubble type investigated by us in the section dedicated to the simple solutions of the double oscillator field theory were studied some time ago by N.A. Voronov and I. Y. Kobzarev and their results were re-asserted in several contemporary papers *[8]*. In particular it was found that these configurations are reasonably long lived[11], namely that these kind of bubbles undergo several pulsations of their radius before decaying into outgoing waves, for instance. Hence, the configurations found by us as possible regular bubbles solutions,

$\Xi_+ = \{(+,+),(+,-)\}$ and $\Xi_- = \{(-,-),(-,+)\}$,

are likely to be "reasonably stable". We cannot say of course too much in this respect as far as the composite solutions are concerned, as no exact identification of them could be produced by using the classical techniques. Nevertheless we do expect approximately the same reasonable stability as the composite character would not influence the undergoing of radius pulsations before decay.

We have investigated in this paper classical solutions to the double oscillator field theory, using non-perturbative methods. Simple solutions to this theory were successfully found. They are regular charge solutions and regular bubble solutions of the form $\Xi_+ = \{(+,+),(+,-)\}$ and $\Xi_- = \{(-,-),(-,+)\}$. It was at the same time proved that second type of regular bubble solutions (of the form $\Xi^+ = \{(+,-),(+,+)\}$ and $\Xi^- = \{(-,+),(-,-)\}$) do not exist as proper solutions to the double oscillator field theory, even if from a sole exterior point of view their existence is not precluded. We have further investigated complex solutions to the field theory analyzing them from the exterior perspective. We found the limiting cases for zero separation between the test charges, respectively infinite separation between the charges, as the analogues of the regular cases. The idea of

[10] The nucleation of the bubbles *[5]*, *[6]* takes place before their per se expansion, but this phenomena is beyond the purpose of the present paper

[11] The Russian team working on this research employed a numerical study of the classical evolution of the field of the bubble-type configuration and has revealed that in the long run the bubbles will emit a large portion of

the regular cases being limits of the composite solutions was thus conveyed. An educated guess on the behavior of the composite solutions near the extrema has been attempted as well.

The paper based itself on the translation of the quantum mechanical double oscillator theory in the scalar fields theory and to this end it made extensive use of the models developed in both these sources. Reviews of major works such as Merzbacher's quantum mechanics or the self-interacting scalar field sources by Consoli and Stevenson were included as being extremely relevant.

In the end, the author hopes that the issue left open, namely the relation between the double oscillator field theory and the $\lambda\phi^4$ theory, will be answered in the near future and thus a further step in understanding quantum field theory would be undertaken.

Acknowledgement*. The author would like to express his gratitude to Dr. Frank Witte for extremely useful discussions and for access to the notes on the subject matter that he previously developed.*

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their energy in outgoing waves. Nonetheless, it was found that these bubbles undergo at least several oscillations before that happens, therefore the label "reasonably long lived" was commonly agreed upon

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