Lecture Notes on Analysis II
MA131

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The lecture notes are based on this and previous years' lectures by myself and by the previous lecturers. I would like to thank David Mond, who is the driving force for me to finish this work and for valuable feed backs. I also like to thank him for providing additional figures. In this edition I would like to thank Ruoran Huang, Chris Midgley, Esther Wershof for a list of typos.

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Contents

The module

0.1 Introduction

Analysis II, together with Analysis I, is a 24 CATS core module for first year students. The content of the module will be delivered in 29 lectures.
Assessment

- 7.5%, Term 1 assignments,
- 7.5%, Term 2 assignments,
- 25%, January exam (on Analysis 1)
- 60%, final exam (3 hour in June, cover analysis 1+2)

Fail the final ===> Fail the module.
Assignments: Given out on Thursday and Fridays in lectures, to hand in the following Thursday in supervisor's pigeonhole by 14:00. Each assignment is divided into part A, part B and part C. Part B is for assignments.

Learn Definitions and statements of theorems. Take the exercises very seriously. It is the real way you learn (and hence pass the summer exam)
Books:
G. H. Hardy: "A Course of Pure Mathematics", Third Edition (First Edition 1908), Cambridge University Press.
E. T. Whittaker &G. N. Watson "A Course of Modern Analysis"
T. M. Apostol "Calculus ", Vol I&II, 2nd Edition, John Wiley & Sons
E. Hairer & G. Wanner: "Analysis by Its History", Springer.
Objectives of the module: Study functions of one real variable. This module leads to analysis of functions of two or more variables (Analysis III), Ordinary differential equations, partial differential equations, Fourier Analysis, Functional analysis etc.
What we cover:

- Continuity of functions of one real variable (lectures 1-3)
  the Intermediate Value theorem, continuous functions on closed intervals
- Continuous limits (circa lecture 8-9)
- Extreme Value Theorem, Inverse Function Theorem (circa 10-11)
- Differentiation (circa lecture 12), Mean Value Theorem
- Power series (ca. lecture 23)
- Taylor's Theorem (ca. lecture 23)
- L'Hôpital's rule (ca. lecture 26)

Relation to Analysis I. Continuity and continuous limits of function will be formulated in terms of sequential limits. Power series are functions obtained by sums of infinite series.

Relation to Analysis II. Uniform convergence and the Fundamental Theorem of Calculus from Analysis III would make a lot of proofs here easier.

Relation to Complex Analysis The power series expansion.

Relation to Ordinary Differential Equations Differentiability of functions. Solve e.g. $f'(x) = f(x), f(0) = 1$.
Feedbacks Ask questions in lectures. Talk to me after or before the lectures.

Plan your study: Shortly after each hour of lecture, you need to spend one hour going over lecture notes and mulling over the assignments. It is a rare student who has understood everything during the lecture. There is plentiful evidence that putting in this effort shortly after the lecture pays ample dividends in terms of understanding. In particular, if you begin a lecture having understood the previous one, you learn much more from it, so the process is cumulative. Please plan two hours per week for the exercises.

Topics by Lecture (approximate dates)

1. Introduction. Continuity, $\epsilon - \delta$ formulation
2. Properties of continuous functions, sequential continuity
3. Algebra of continuity
4. Composition of continuous functions, examples
5. The Intermediate Value Theorem (IVT)
6. The Intermediate Value Theorem (continued)
7. Continuous Limits, $\epsilon - \delta$ formulation, relation with to sequential limits and continuity
8. One sided limits, left and right continuity
9. The Extreme Value Theorem
10. The Inverse Function Theorem (continuous version)
11. Differentiation, Weierstrass formulation
12. Algebraic rules, chain rule
13. The Inverse Function Theorem (Differentiable version)
14. Local Extrema, critical points, Rolle's Theorem
15. The Mean Value Theorem
16. Deduce Inequalities, asymptotics of f at infinity
17. Higher order derivatives, $C^k(a, b)$ functions, examples of functions, convexity and graphing
18. Formal power series, radius of convergence
19. Limit superior,
20. Hadamard Test for power series, Functions defined by power series
21. Term by Term Differentiation Theorem
22. Classical Functions of Analysis

23. Polynomial approximation, Taylor's series, Taylor's formula
24. Taylor's Theorem
25. Cauchy's Mean Value Theorem, Techniques for evaluating limits
26. L'Hôpital's rule
27. L'Hôpital's rule
28. Unfinished business, Summary, Matters related to exams
29. Question Time, etc

Chapter 1

Continuity of Functions of One Real Variable

Let $\mathbf{R}$ be the set of real numbers. Let E denote a subset of $\mathbf{R}$. Here are some examples of the kind of subsets we will be considering: $E = \mathbf{R}$, $E = (a, b)$ (open interval), $E = [a, b]$ (closed interval), $E = (a, b]$ (semi-closed interval), $E = [a, b)$, $E = (a, \infty)$, $E = (-\infty, b)$, and $E = (1, 2) \cup (2, 3)$. The set of rational numbers $\mathbf{Q}$ is also a subset of $\mathbf{R}$.

Definition 1.1 * *A point x_0 is called an accumulation point of a set A if for any $\delta > 0$, there is a point $x \in A$ such that $0 < |x - x_0| < \delta$.*

Example 1.1 *

- If E is the union of a finite number of intervals, any point in E is an accumulation point.
- The end point of an open interval is an accumulation point.
- $\{\frac{1}{n} : n \in \mathbf{N}_{>0}\} \subset \mathbf{R}$ has only one accumulation point: $\{0\}$
- $\{1, 2, \ldots, \}$ has no accumulation points.
- every real number is an accumulation point of $\mathbf{Q}$.
- every real number is an accumulation point of $\mathbf{R} - \mathbf{Q}$.

1.1 Functions – Lecture 1

Definition 1.2 *By a function $f : E \to \mathbf{R}$ we mean a rule which to every number in E assigns a number from $\mathbf{R}$. This correspondence is denoted by*

$$y = f(x), \qquad or \qquad x \mapsto f(x).$$

- *We say that y is the image of x and x is a pre-image of y.*
- *The set E is the domain of f.*
- *The range of f consists of the images of all points of E. It is often denoted $f(E)$.*

We denote by $\mathbf{N}$ the set of natural numbers, $\mathbf{Z}$ the set of integers and $\mathbf{Q}$ the set of rational numbers:

$$\begin{aligned} \mathbf{N} &= \{1, 2, \dots\} \\ \mathbf{Z} &= \{0, \pm 1, \pm 2, \dots\} \\ \mathbf{Q} &= \{\frac{p}{q} : p, q \in Z, q \neq 0\}. \end{aligned}$$

Example 1.2

1. $E = [-1, 3]$.

$$f(x) = \begin{cases} 2x, & -1 \leq x \leq 1 \\ 3 - x, & 1 < x \leq 3. \end{cases}$$

The range of f is $[-2, 2]$.

2. $E = \mathbf{R}$.

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbf{Q} \\ 1, & \text{if } x \not\in \mathbf{Q} \end{cases}$$

Range[f]$= \{0, 1\}$.

3. $E = \mathbf{R}$.

$$f(x) = \begin{cases} 1/q, & \text{if } x = \frac{p}{q} \text{ where } p \in \mathbf{Z}, q \in \mathbf{N} \text{ have no common divisor} \\ 0, & \text{if } x \not\in \mathbf{Q} \end{cases}$$

Range[f]$= \{0\} \cup \{\frac{1}{q}, q = \pm 1, \pm 2, \dots\}$.

4. $E = R$.

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

Range[f]$= [c, 1]$ where $c = \min \frac{\sin x}{x}$.

5. $E = \mathbf{R}$.
$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$$
Range[f]= $[c, 1) \cup \{2\}$.

6. $E = (-1, 1)$.
$$f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}.$$
This function is a representation of $-\log(1+x)$, see chapter on Taylor series. Range[f]= $(-\log 2, \infty)$.

1.2 Useful Inequalities and Identities

$$\begin{aligned} a^2 + b^2 &\geq 2ab \\ |a+b| &\leq |a| + |b| \\ |b-a| &\geq \max(|b| - |a|, |a| - |b|). \end{aligned}$$

Proof The first follows from $(a-b)^2 > 0$. The second inequality is proved by squaring $|a+b|$ and noting that $ab \leq |a||b|$. The third inequality follows from the fact that
$$|b| = |a + (b-a)| \leq |a| + |b-a|$$
and by symmetry $|a| \leq |b| + |b-a|$.

$\square$

Pythagorean theorem : $\sin^2 x + \cos^2 x = 1$.
$$\cos^2 x - \sin^2 x = \cos(2x), \qquad \cos(x) = 1 - 2\sin^2(\frac{x}{2}).$$

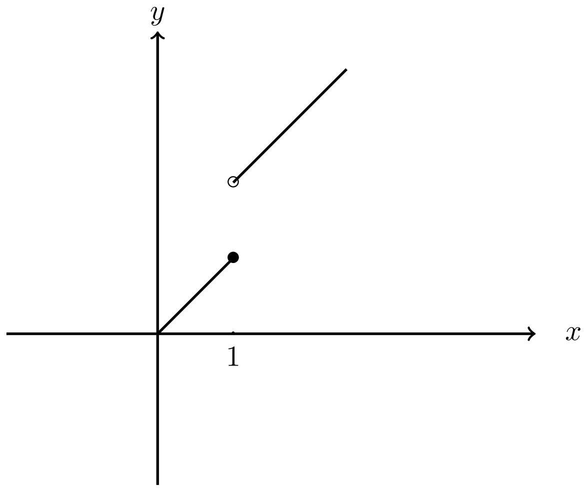
1.3 Continuous Functions – Lecture 2

What do we mean by saying that "$f(x)$ varies continuously with x"?

It is reasonable to say f is continuous if the graph of f is an unbroken continuous curve. The concept of an unbroken continuous curve seems easy to understand. However we may need to pay attention.

For example we look at the graph of
$$f(x) = \begin{cases} x, & x \leq 1 \\ x+1, & \text{if } x > 1 \end{cases}$$

It is easy to see that the curve is continuous everywhere except at $x = 1$. The function is not continuous at $x = 1$ since there is a gap 1 between the values of $f(1)$ and $f(x)$ for x close to 1. It is continuous everywhere else.



Now take the function

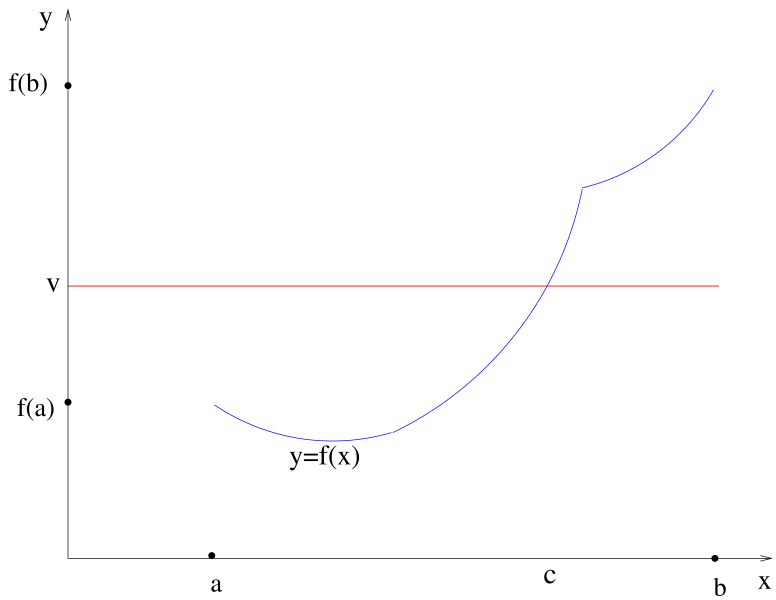
$$F(x) = \begin{cases} x, & x \leq 1 \\ x + 10^{-30}, & \text{if } x > 1 \end{cases}$$

The function is not continuous at $x = 1$ since there is a gap of 10^{-30}. However can we see this gap on a graph with our naked eyes? No, unless you have exceptional eyesight!

Here is a theorem we will prove, once we have the definition of "continuous function".

Theorem 1.1 *(Intermediate value theorem): Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Suppose that $f(a) \neq f(b)$, and that the real number v lies between $f(a)$ and $f(b)$. Then there is a point $c \in [a, b]$ such that $f(c) = v$.*

This looks "obvious", no? In the picture shown here, it says that if the graph of the continuous function $y = f(x)$ starts, at $(a, f(a))$, *below* the straight line $y = v$ and ends, at $(b, f(b))$, *above* it, then at some point between these two points it must *cross* this line.

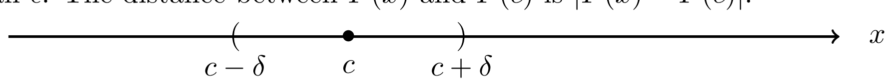


But how can we *prove* this? Notice that its truth uses some subtle facts about the real numbers. If, instead of the domain of f being an interval in $\mathbf{R}$, it is an interval in $\mathbf{Q}$, then the statement is no longer true. For example, we would probably agree that the function $F(x) = x^2$ is "continuous" (soon we will see that it is). If we now consider the function $f : \mathbf{Q} \to \mathbf{Q}$ defined by the same formula, then the rational number $v = 2$ lies between $0 = f(0)$ and $9 = f(3)$, but even so there is no c between 0 and 3 (or anywhere else, for that matter) such that $f(c) = 2$.

Sometimes what seems obvious becomes a little less obvious if you widen your perspective.

These examples call for a proper definition for the notion of continuous function.

We use the letter ϵ to bound the size of the gap. Having no gap should be interpreted as for any $\epsilon > 0$, the distance between $F(x)$ and $F(c)$ is smaller than ϵ. The distance between $F(x)$ and $F(c)$ is $|F(x) - F(c)|$.



Important: The distance between x and c is $|x - c|$. The set of x such that $|x - c| < \delta$ is the same as the set of x with $c - \delta < x < c - \delta$.

Let us consider again the function f

$$f(x) = \begin{cases} x, & x \leq 1 \\ x+1, & \text{if } x \geq 1 \end{cases}$$

Is it continuous at $c = 1 - 2^{-200}$? The answer is yes. The value $f(x) - f(c)$ is very small for x sufficiently close to c. How close need it be? We may require that $|x - c| < 2^{-200}$ (For such x, $x \le 1$ and $f(x) = 2$). The number which describes the closeness of x to c is usually denoted by δ.

Definition 1.3 *Let E be subset of $\mathbf{R}$ and c a point of E.*

1. *A function $f : E \to \mathbf{R}$ is continuous at c if for every $\epsilon > 0$ there exists a $\delta > 0$ such that*
$$\begin{array}{ll} \textit{if} & |x - c| < \delta \ \textit{and} \ x \in E \\ \textit{then} & |f(x) - f(c)| < \epsilon. \end{array}$$

2. *If f is continuous at every point c of E, we say f is continuous on E or simply that f is continuous.*

The reason that we request that $x \in E$ is that f is only defined for points in E! If f is a function with domain $\mathbf{R}$, we may drop E in the formulation above.

Example 1.3 Show that $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2 + x$ is continuous at $x = 2$.

Solution. For an arbitrary $\epsilon > 0$ take $\delta = \min(\frac{\epsilon}{6}, 1)$. Then if $|x - 2| < \delta$,

$$|x| \le |x - 2| + 2 \le \delta + 2 \le 1 + 2 = 3.$$

And $|x + 2| \le 5$,

$$\begin{aligned} |f(x) - f(2)| &= |x^2 + x - (4 + 2)| \le |x^2 - 4| + |x - 2| \\ &= |x + 2||x - 2| + |x - 2| \le 5|x - 2| + |x - 2| \\ &= 6|x - 2| < 6\delta \\ &\le \epsilon. \end{aligned}$$

Exercise 1.4 Show that the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ given by $g(x) = 2x$ and $f(x) = 3 - x$ are continuous.

Let us now do a logic problem. We defined what is meant by 'f is continuous at c'. How do we formulate 'f is not continuous at c'?

Remark 1.1 ([use of negation : 'f is not continuous at c') A function $f : E \to \mathbf{R}$ is not continuous at c, otherwise known as discontinuous at c, precisely means that:

There exists a number $\epsilon > 0$ such that for all $\delta > 0$ there exists a $x \in E$ with

$$|x - c| < \delta, \qquad |f(x) - f(c)| \geq \epsilon.$$

Example 1.5 Consider $f : \mathbf{R} \to \mathbf{R}$,

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbf{Q} \\ 1, & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Claim: The function is not continuous at any point.

The key point is that between any two numbers there are a rational number and an irrational number.

Case 1. If $c \in \mathbf{Q}$, let us take $\epsilon = 0.5$. For any $\delta > 0$, take $x \notin Q$ with $|x - c| < \delta$ then $|f(x) - f(c)| = |1 - 0| > 0.5$.

Case 2. If $c \notin \mathbf{Q}$ we also take $\epsilon = 0.5$. For any $\delta > 0$, take $x \in Q$ with $|x - c| < \delta$ then $|f(x) - f(c)| = |0 - 1| > 0.5$.

Example 1.6 Consider $f : \mathbf{R} \to \mathbf{R}$,

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbf{Q} \\ x, & \text{if } x \notin \mathbf{Q} \end{cases}$$

Claim: This function is continuous at $c = 0$ and discontinuous everywhere else.

If $c = 0$. For any $\epsilon > 0$ take $\delta = \epsilon$. If $|x - c| < \delta$, then $|f(x) - f(0)| = |x|$ in case $c \notin \mathbf{Q}$, and $|f(x) - f(0) = |0 - 0| = 0$ in case $c \in \mathbf{Q}$. In both cases, $|f(x) - f(0)| \leq |x| < \delta = \epsilon$. Hence f is continuous at 0.

Show that f is not continuous at $c \neq 0$! [Hint: take $\epsilon = |x|/2$.]

Exercise 1.7 Suppose that $g : \mathbf{R} \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$ are continuous. We may define a new function $f : \mathbf{R} \to \mathbf{R}$

$$f(x) = \begin{cases} g(x), & x \in \mathbf{Q} \\ h(x), & x \notin \mathbf{Q}. \end{cases}$$

Discuss the continuity of the function f. [Hint consider all points x with $g(x) = h(x)$.]

Remark 1.2 Suppose that $g : \mathbf{R} \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$ are continuous. We may define a new function $f : \mathbf{R} \to \mathbf{R}$

$$f(x) = \begin{cases} g(x), & x < c \\ h(x), & x \geq c. \end{cases}$$

Then the function f is continuous at c if and only if $g(c) = h(c)$.

Proof Since g and h are continuous at c, for any $\epsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that $|g(x) - g(c)| < \epsilon$ when $|x - c| < \delta_1$ and such that $|h(x) - h(c)| < \epsilon$ when $|x - c| < \delta_2$. Define $\delta = \min(\delta_1, \delta_2)$. Then if $|x - c| < \delta$,

$$|g(x) - g(c)| < \epsilon, \qquad |h(x) - h(c)| < \epsilon.$$

- Case 1: Suppose $g(c) = h(c)$. For any $\epsilon > 0$ and the above δ, if $|x - c| < \delta$,

$$|f(x) - f(c)| = \begin{cases} |g(x) - g(c)| < \epsilon & \text{if } x < c \\ |h(x) - h(c)| < \epsilon & \text{if } x \geq c \end{cases}$$

  So f is continuous at c.

- Case 2: Suppose $g(c) \neq h(c)$.

  Take

$$\epsilon = \frac{1}{2}|g(c) - h(c)|.$$

  By the continuity of g at c, there exists δ_0 such that if $|x - c| < \delta_0$, then $|g(x) - g(c)| < \epsilon$. If $x < c$, then

$$|f(c) - f(x)| = |h(c) - g(x)|;$$

  moreover

$$|h(c) - g(x)| = |h(c) - g(c) + g(c) - g(x)| \geq |h(c) - g(c)| - |g(c) - g(x)|.$$

  We have $|h(c) - g(c)| = 2\epsilon$, and if $|c - x| < \delta_0$ then $|g(c) - g(x)| < \epsilon$. It follows that if $c - \delta_0 < x < c$,

$$|f(c) - f(x)| = |h(c) - h(x)| > \epsilon.$$

  So if $c - \delta_0 < x < c$, then no matter how close x is to c we have

$$|f(x) - f(c)| > \epsilon.$$

  This shows f is not continuous at c.

$\square$

Example 1.8 Let $E = [-1, 3]$. Consider $f : [-1, 3] \to \mathbf{R}$,

$$f(x) = \begin{cases} 2x, & -1 \leq x \leq 1 \\ 3 - x, & 1 < x \leq 3. \end{cases}$$

The function is continuous everywhere.

A rational number in $\mathbf{Q} \cap (0,1)$ can be expressed by p/q where $p, q \in \mathbf{N}$. We assume that p and q have no common factor.

Example 1.9 Let $f : (0,1) \to \mathbf{R}$.

$$f(x) = \begin{cases} 1/q, & \text{if } x = \frac{p}{q}, p < q, \quad p, q \in \mathbf{N} \\ 0, & \text{if } x \notin \mathbf{Q} \end{cases}$$

Then f is continuous on all irrational points, and discontinuous on all rational points.

Proof

- Case 1. We show that f is not continuous if $c \in Q$. If $c = \frac{p}{q}$, take $\epsilon = \frac{1}{2q}$. No matter how small is δ there is an irrational number x such that $|x - c| < \delta$. And $|f(x) - f(c)| = |\frac{1}{q}| > \epsilon$.
- Case 2. Let $c \notin Q$. Take an arbitrary positive number ϵ. We estimate $|f(x) - f(c)|$. Furthermore if $x \notin Q$, then $|f(x) - f(c) = |0 - 0| = 0$. If $x = \frac{p}{q}$, $|f(x) - f(c)| = \frac{1}{q}$. For which $x = \frac{p}{q}$, $\frac{1}{q} > \epsilon$?
  - There is a finite number of natural numbers q such that $q < \frac{1}{\epsilon}$.
  - Let A be the set of rational $\frac{p}{q}$ satisfying that $p < q$ and $q < \frac{1}{\epsilon}$. There are only finitely many elements in the set A.

  Choose δ small so that $(c - \delta, c + \delta)$ does not contain any number from A. Hence if $|x - c| < \delta$, $x = \frac{p}{q} \in Q$ then $|f(x) - f(c)| = |\frac{1}{q}| < \epsilon$.

$\square$

Example 1.10 $f : (-1,1) \to \mathbf{R}$.

$$f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$$

Is f continuous at 0? This is a difficult problem for us at the moment. We cannot even graph this function and so we have little intuition. But we can answer this question after the study of power series.

Exercise 1.11 Is there a value a such that the function

$$f_a(x) = \begin{cases} 1, & \text{if } x > 0 \\ a, & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases}$$

is continuous at $x = 0$? Justify your answer?

1.4 Continuity and Sequential Continuity –Lecture 3

Definition 1.4 *We say $f : E \to \mathbf{R}$ is sequentially continuous at $c \in E$ if for every sequence $\{x_n\} \subset E$ such that $\lim_{n\to\infty} x_n = c$ we have*

$$\lim_{n\to\infty} f(x_n) = f(c).$$

Theorem 1.2 *Let E be a subset of $\mathbf{R}$ and $c \in E$. Let $f : E \to \mathbf{R}$. The following are equivalent:*

1. *f is continuous at c.*
2. *f is sequentially continuous at c.*

Proof

- Suppose that f is continuous at c. For any $\epsilon > 0$ there is a $\delta > 0$ so that
$$|f(x) - f(c)| < \epsilon, \qquad \text{whenever } |x - c| < \delta.$$
Let $\{x_n\}$ be a sequence with $\lim_{n\to\infty} x_n = c$. There is an integer N such that
$$|x_n - c| < \delta, \qquad \forall n > N,$$
Hence if $n > N$,
$$|f(x_n) - f(c)| < \epsilon.$$
We have proved that f is sequentially continuous at c.

- Suppose that f is not continuous at c.
Then $\exists \epsilon > 0$ such that for all $\delta > 0$, there is a point $x \in E$ with $|x - c| < \delta$ but $|f(x) - f(c)| \not< \epsilon$.
Choose $\delta = \frac{1}{n}$. Then there exists x_n with $|x_n - c| < \frac{1}{n}$, with
$$|f(x_n) - f(c)| \geq \epsilon.$$
In this way we have constructed a sequence $\{x_n\}$ for which the the statement '$\lim_{n\to\infty} f(x_n) = f(c)$' does not hold.
But $|x_n - c| < \frac{1}{n}$ implies that $\lim_{n\to\infty} x_n = c$. We have proved that f is not sequentially continuous at c if f is not continuous at c.

$\square$

Sequential continuity is handy when it comes to showing a function is discontinuous.

Example 1.12 Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not continuous at 0.
Proof Observe that $f(0) = 0$. Take $x_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ then $x_n \to 0$ as $n \to \infty$. But

$$f(x_n) = 1 \not\to 0.$$

Hence f is not sequentially continuous at 0 and it is therefore not continuous at 0. $\square$

Example 1.13 Let

$$f(x) = \begin{cases} x+1 & x \neq 2 \\ 12 & x = 2 \end{cases}$$

The function f is not continuous at $x = 2$.
Proof Take $x_n = 2 + \frac{1}{n}$; then $x_n \to 2$ as $n \to \infty$, but

$$f(x_n) = 3 + \frac{1}{n} \to 3 \neq f(2).$$

Hence f is not sequentially continuous at 2 and it is therefore not continuous at 2.

$\square$

1.5 Properties of Continuous Functions – Lecture 4

In this section we learn how to build continuous functions from known continuous functions.

We recall the following properties of sequential limits:

Proposition 1.3 (Algebra of Limits) *Suppose a_n and b_n are sequences with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then*

- $\lim_{n\to\infty}(a_n + b_n) = a + b$;
- $\lim_{n\to\infty}(a_n b_n) = ab$;
- $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{a}{b}$, *provided that* $b_n \neq 0$ *for all* n *and* $b \neq 0$.

What can we deduce about the continuity of $f+g$, fg and f/g if f and g are continuous? Let us first show that if f is continuous at c and $f(c) \neq 0$ then there is a non-trivial interval containing c on which f does not vanish.

For $r > 0$, let

$$B_r(c) = \{x \in E \,| c - r < x < c + r\}.$$

This is the subset of E whose elements satisfy $c - r < x < c + r$. We say $B_r(c)$ a *neighbourhood of* c *in* E.

The following lemma says that if f is continuous at c and $f(c) \neq 0$ then f does not vanish close to c. In fact f has the same sign as $f(c)$ in a neighbourhood of c.

Lemma 1.4 *[Non-vanishing lemma] Suppose that* $f : E \to \mathbf{R}$ *is continuous at* c.

1. *If* $f(c) > 0$, *then there exists* $r > 0$ *such that* $f(x) > 0$ *for* $x \in B_r(c)$.
2. *If* $f(c) < 0$, *then* $\exists\ r > 0$ *s.t.* $f(x) < 0$ *if* $x \in B_r(c)$.

In both cases there is a neighbourhood of c *on which* f *does not vanish.*

Proof Suppose that $f(c) > 0$. Take $\epsilon = \frac{f(c)}{2} > 0$. Let $r > 0$ be a number such that if $|x - c| < r$ and $x \in E$ then

$$|f(x) - f(c)| < \epsilon.$$

For such x, $f(x) > f(c) - \epsilon = \frac{f(c)}{2} > 0$.

If $f(c) < 0$, take $\epsilon = \frac{-f(c)}{2} > 0$. There is $r > 0$ such that on $B_r(c)$, $f(x) < f(c) + \epsilon = \frac{f(c)}{2} < 0$. $\square$

Exercise 1.14 Let f be continuous at c. Show that if $v \in \mathbf{R}$ and $f(x) < v$ then there exists $\delta > 0$ s.t. for all $x \in (c - \delta, c + \delta)$, $f(x) < v$.

Proposition 1.5 *[Algebra of continuity] Suppose* f *and* g *are both defined on a subset* E *of* $\mathbf{R}$ *and are both continuous at* c, *then*

1. *$af + bg$ is continuous at c where a and b are any constants.*

2. *fg is continuous at c*

3. *f/g is continuous at c if $g(c) \neq 0$.*

Proof Let $\{x_n\}$ be any sequence in E converging to c. Then

$$\lim_{n\to\infty} f(x_n) = f(c), \qquad \lim_{n\to\infty} g(x_n) = g(c).$$

- By the algebra of convergent sequences we see that

  $$\lim_{n\to\infty}[af(x_n) + bg(x_n)] = [af(c) + bg(c)].$$

  Thus

  $$\lim_{n\to\infty}(af + bg)(x_n) = (af + bg)(c).$$

  Hence $af + bg$ is sequentially continuous at c.

- Since $\lim_{n\to\infty} f(x_n)g(x_n) = f(c)g(c)$,

  $$\lim_{n\to\infty}(fg)(x_n) = (fg)(c)$$

  and fg is sequentially continuous at c.

- Suppose that $g(c) \neq 0$. By Lemma 1.4, there is a neighbourhood $B_r(c)$ such that $g(x) \neq 0$. We may now assume that $E = B_r(c)$ and hence assume that $x_n \in B_r(c)$.

  Any sequences in $B_r(c)$ satisfies $g(x_n) \neq 0$. By sequential continuity of g, $\lim_{n\to\infty} g(x_n) = g(c) \neq 0$. Consequently

  $$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{f(c)}{g(c)}$$

  and $\frac{f}{g}$ is sequentially continuous at c.

$\square$

The continuity of the function $f(x) = x^2 + x$, which we proved from first principles in Example 1.3, can be proved more easily using Proposition 1.5. For the function $g(x) = x$ is obviously continuous (take $\delta = \epsilon$), the function $h(x) = x^2$ is equal to $g \times g$ and is therefore continuous by 1.5(2), and finally $f = h + g$ and is therefore continuous by 1.5(1).

Example 1.15 The function f given by $f(x) = \frac{1}{x}$ is continuous on $\mathbf{R} - \{0\}$. It cannot be extended to a continuous function on $\mathbf{R}$ since if $x_n = \frac{1}{n}$, $f(x_n) \to \infty$.

Proposition 1.6 *[composition of functions] Suppose $f : E \to \mathbf{R}$ is continuous at c, g is defined on the range of f, and g is continuous at $f(c)$. Then $g \circ f$ is continuous at c.*

Proof Let $\{x_n\}$ be any sequence in E converging to c. By the continuity of f at c, $\lim_{n\to\infty} f(x_n) = f(c)$. By the continuity of g at $f(c)$,

$$\lim_{n\to\infty} g(f(x_n)) = g(f(c)).$$

This shows that $g \circ f$ is continuous at c. $\square$

Example 1.16

1. A polynomial of degree n is a function of the form $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Polynomials are continuous, by repeated application of Proposition 1.5(1) and (2).

2. A *rational function* is a function of the form: $\frac{P(x)}{Q(x)}$ where P and Q are polynomials. The rational function $\frac{P}{Q}$ is continuous at x_0 provided that $Q(x_0) \neq 0$, by 1.5(3).

Example 1.17

1. The exponential function $\exp(x) = e^x$ is continuous. This we will prove later in the course.

2. Given that exp is continuous, the function g defined by $g(x) = \exp(x^{2n+1} + x)$ is continuous (use 1 above, 1.16(1) and 1.6).

Example 1.18 The function $x \mapsto \sin x$ is continuous. This will be proved later using the theory of power series. An alternative proof is given in the next section which is not covered in lectures. From the continuity of $\sin x$ we can deduce that $\cos x$, $\tan x$ and $\cot x$ are continuous:

Example 1.19 The function $x \mapsto \cos x$ is continuous by 1.6, since $\cos x = \sin(x + \frac{\pi}{2})$ and is thus the composite of the continuous function cos and the continuous function $x \mapsto x + \pi/2$

Example 1.20 The function $x \mapsto \tan x$ is continuous. Use $\tan x = \frac{\sin x}{\cos x}$ and 1.5(3).

Discussion on $\tan x$**.** If we restrict the domain of $\tan x$ to the region] $(-\frac{\pi}{2}, \frac{\pi}{2})$, its graph is a continuous curve and we believe that $\tan x$ is continuous. On a larger range, $\tan x$ is made of disjoint pieces of continuous curves. How could it be a continuous function ? Surely the graph looks discontinuous at $\frac{\pi}{2}$!!! The trick is that the domain of the function does not contain these points where the graph looks broken. By the definition of continuity we only consider x with values in the domain.

The largest domain for $\tan x$ is

$$\mathbf{R}/\{\frac{\pi}{2} + k\pi, k \in Z\} = \cup_{k\in Z}(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}).$$

For each c in the domain, we locate the piece of continuous curve where it belongs. We can find a small neighbourhood on which the graph is is part of this single piece of continuous curve.

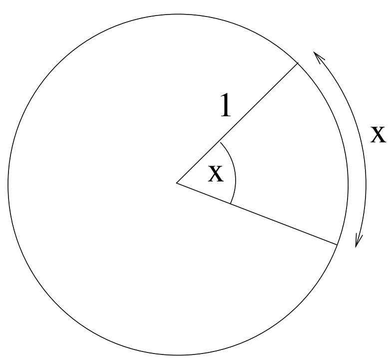
For example if $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$, make sure that δ is smaller than $\min(\frac{\pi}{2} - c, c + \frac{\pi}{2})$.

1.6 Continuity of Trigonometric Functions*

This section is not covered in lectures.

About angles.

Two different units are commonly used for measuring angles: Babylonian degrees and radians. We will use radians. The radian measure of an angle x is the length of the arc of a unit circle subtended by the angle x.



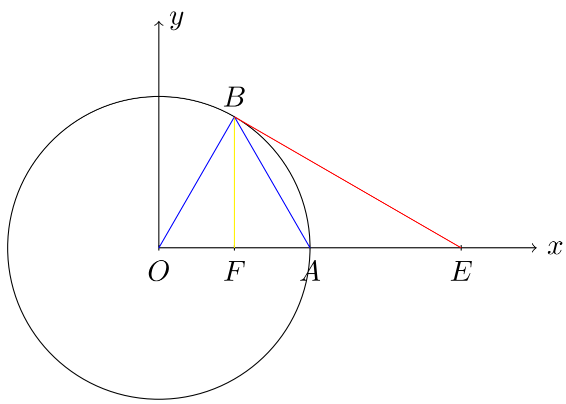
The following explains how to measure the length of an arc. Take a polygon of n segments of equal length inscribed in the unit circle. Let ℓ_n be its length. The length increases with n. As $n \to \infty$, it has a limit. The limit is called the *circumference* of the unit circle. The circumference of the unit circle was measured by Archimedes, Euler, Liu Hui etc.. It can now be shown to be an irrational number.

Historically $\sin x$ and $\cos x$ are defined in the following way. Later we define them by power series. The two definitions agree. Take a right-angled triangle, with the hypotenuse of length 1 and an angle x. Define $\sin x$ to be the length of the side facing the angle and $\cos x$ the length of the side adjacent to it. Extend this definition to $[0, 2\pi]$. For example, if $x \in [\frac{\pi}{2}, \pi]$, define $\sin x = \cos(x - \frac{\pi}{2})$.

Lemma 1.7 *If* $0 < x < \frac{\pi}{2}$, *then*

$$\sin x \le x \le \tan x.$$

Proof Take a unit circle centred at 0, and consider a sector of the circle of angle x which we denote by OBA.



The area of the sector OBA is $x/2pi$ times the area of the circle, and is therefore $\pi\frac{x}{2\pi} = \frac{x}{2}$. The area of the triangle OBA is $\frac{1}{2}\sin x$. So

$$\sin x \le x.$$

Consider the right-angled triangle OBE, with one side tangent to the circle at B. Because $BE = \tan x$, the area of the triangle is $\frac{1}{2}\tan x$. This triangle contains the sector OBA. So

$$\text{Area(Sector } OBA) \;\le\; \text{Area(Triangle } OBE),$$

and therefore $x \le \tan x$.

□

Theorem 1.8 *The function $x \mapsto \sin x$ is continuous.*

Proof

$$
\begin{aligned}
|\sin(x+h) - \sin x| &= |\sin x \cos h + \cos x \sin h - \sin x| \\
&= |\sin x(\cos h - 1) + \cos x \sin h| \\
&\le 2|\sin x||\sin^2 \frac{h}{2}| + |\cos x||\sin h| \\
&\le |h^2| + |h| = |h|(|h| + 1).
\end{aligned}
$$

If $|h| \le 1$, then $(\frac{|h|}{2} + 1) < \frac{3}{2}$. For any $\epsilon > 0$, choose $\delta = \min(\frac{2}{3}\epsilon, 1)$. If $|h| < \delta$ then

$$
|\sin(x+h) - \sin x| = |h|(\frac{|h|}{2} + 1) \le \frac{3}{2}|h| < \epsilon.
$$

$\square$

Chapter 2

Continuous Functions on closed Intervals: I

We recall the definition of the least upper bound or the supremum of a set A.

Definition 2.1 *A number c is an upper bound of a set A if for all $x \in S$ we have $x \leq c$. A number c is a least upper bound of a set A if*

- *c is an upper bound for A.*
- *if U is an upper bound for A then $c \leq U$.*

This number is denoted by $\sup A$.

The completeness Axiom If $S \subset \mathbf{R}$ is a non empty set bounded above it has a least upper bound in $\mathbf{R}$.

In particular there is a sequence $x_n \in S$ such that $c - \frac{1}{n} \leq x_n < c$. This sequence x_n converges to c.

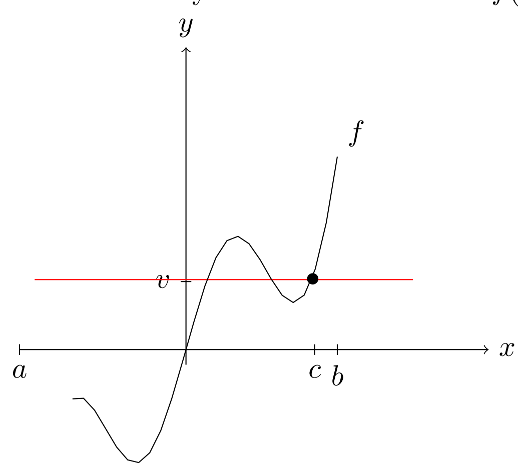
2.1 The Intermediate Value Theorem – Lectures 5 & 6

Recall Theorem 1.1, which we state again:

Theorem 2.1 (The Intermediate Value Theorem(IVT)) *Let $f : [a, b] \to \mathbf{R}$ be continuous. Suppose that $f(a) < f(b)$. Then for any v with $f(a) < v < f(b)$ there exists $c \in (a, b)$ such that $f(c) = v$.*

A rigorous proof was first given by Bolzano 1817.

Proof Consider the set $A = \{x \in [a,b] : f(x) \le v\}$. Note that $a \in A$ hence A is not empty and A is bounded above by b, so it has a least upper bound, which we denote by c. We will show that $f(c) = v$.



- Since $c = \sup A$, there exists $x_n \in A$ with $x_n \to c$. Then $f(x_n) \to f(c)$ by continuity of f. Since $f(x_n) \le v$ then $f(c) \le v$.

- Suppose that $f(c) \ne v$ then $f(c) < v$ by the previous step. In particular $c \ne b$. Then there exists $0 < r < b - c$ such that for all $x \in [a,b]$ with $c - r < x < c + r$, $f(x) < v$ (Apply the Non-vanishing Lemma 1.4 to the function $x \mapsto v - f(x)$.) It follows that $c + \frac{r}{2} \in [a,b]$ and $c + \frac{r}{2} \in A$, contradicting with the assumption that $c = \sup(A)$, the *least* upper bound for A. Hence $f(c)$ must equal to v.

$\square$

The idea of the proof is to identify the greatest number in (a,b) such that $f(c) = v$. The existence of the least upper bound (i.e. the completeness axiom) is crucial here, as it is on practically every occasion on which one wants to prove that there exists a point with a certain property, without knowing at the start where this point is.

Example 2.1 Let $f : [0,1] \to \mathbf{R}$ be given by $f(x) = x^7 + 6x + 1, x \in [0,1]$. Can we solve $f(x) = 2$ for some $x \in [0,1]$?

Proof The answer is yes. Since $f(0) = 1, f(1) = 8$ and $2 \in (1,8)$, there is a number $c \in [0,1]$ such that $f(c) = 2$ by the IVT. $\square$

Remark*: There is an alternative proof for the IVT, using the "bisection method". It is not covered in lectures. Let f be a continuous function on $[a, b]$ and we may assume that $f(a) < f(b)$. Let $v \in (f(a), f(b))$. By defining a new continuous function $g(x) = f(x) - v$, if necessary, we only need to prove the case of $v = 0$, $f(a) < 0$ and $f(b) > 0$. We show that there is $c \in (a, b)$ with $f(c) = 0$.

We construct nested intervals $[a_n, b_n]$ with length decreasing to zero: $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \ldots$. We then show that a_n and b_n have common limit c which satisfy $f(c) = 0$.

Divide the interval $[a, b]$ into two equal halves: $[a, c_1], [c_1, b]$. If $f(c_1) = 0$, done. If not on (at least) one of the two sub-intervals, the value of f must change from negative to positive. Call this subinterval $[a_1, b_1]$. More precisely, if $f(c_1) > 0$ write $a = a_1$, $b_1 = c_1$; if $f(c_1) < 0$, let $a_1 = c_1$, $b_1 = b$. Iterate this process. Either at the k'th stage $f(c_k) = 0$, or we obtain a sequence of intervals $[a_k, b_k]$ with $[a_{k+1}, b_{k+1}] \subset [a_k, b_k]$, with a_k increasing, b_k decreasing, and $f(a_k) < 0$, $f(b_k) > 0$ for all k. Because the sequences a_k and b_k are monotone and bounded, both converge, say to c' and c'' respectively. We must have $c' \leq c''$, and $f(c') \leq 0 \leq f(c'')$. If we can show that $c' = c''$ then since $f(c') \leq 0$, $f(c'') \geq 0$ then we must have $f(c') = 0$, and we have won. It therefore remains only to show that $c' = c''$. I leave this as an (easy!) exercise.

Theorem 2.2 (The Intermediate Value Theorem–Double Sided Version) *Let $f : [a, b] \to \mathbf{R}$ be continuous. Suppose that $f(a) \neq f(b)$. Then for any v between $f(b)$ and $f(a)$ there exists $c \in (a, b)$ such that $f(c) = v$.*

Proof If $f(a) < f(b)$, this has been proved. If $f(a) > f(b)$, take v with $f(b) < v < f(a)$. Take $g(x) = -f(x)$. Then $g(a) < g(b)$. If $f(b) < v < f(a)$ then $(-f(a) < -v < -f(b))$ and $g(a) < -v < g(b))$. Apply the intermediate value theorem: there exists a point $c \in (a, b)$ with $g(c) = -v$. Thus $f(c) = -g(c) = v$. □

Remark 2.1 The following statements are equivalent:

1) The intermediate value theorem.

2) If $f : [a, b] \to \mathbf{R}$ is continuous with $f(a) < 0$ and $f(b) > 0$ then there exists c such that $f(c) = 0$.

Proof Statement 2 is a special case of the IVT. Supposing that statement 2) holds, we prove that the IVT follows.

Suppose that $f(a) < f(b)$. Take $f(a) < v < f(b)$ Set $g(x) = f(x) - v$. Then $g(a) < 0$ and $g(b) > 0$. By 2) there exists a point $c \in (a, b)$ with $g(c) = 0$. For this c, $f(c) = g(c) + v = v$.

$\square$

Example 2.2 Show that $3x^5 + 5x + 7 = 0$ has a real root.

Proof Let $f(x) = 3x^5 + 5x + 7$. Consider f as a function on $[-1, 0]$. Then f is continuous and $f(-1) = -3 - 5 + 7 = -1 < 0$ and $f(0) = 7 > 0$. By the IVT there is $c \in (-1, 0)$ such that $f(c) = 0$. $\square$

Exercise 2.3 Show that every polynomial of odd degree has a real root.

Discussion on assumptions. In the following examples the statement fails. For each one, which condition required in the IVT is not satisfied?

Example 2.4

1. Let $f : [-1, 1] \to \mathbf{R}$,
$$f(x) = \begin{cases} x + 1, & x > 0 \\ x, & x < 0 \end{cases}$$
Then $f(-1) = -1 < f(1) = 2$. Can we solve $f(x) = 1/2 \in (-1, 2)$? No. Note that the function is not continuous on $[-1, 1]$.

2. Define $f : Q \cap [0, 2] \to \mathbf{R}$ by $f(x) = \sqrt{x}$. $f(0) = 0$ and $f(2) = 2$. Does there exist a number $c \in Q \cap [0, 1]$ such that $f(c) = v$ any $v \in [0, 1]$? No. Take $v = \sqrt{2} \in (0, 2)$. But f is not defined at $\sqrt{2}$.

   Note that the domain of f is $\mathbf{Q}$, not an interval.

Example 2.5 The Intermediate Value Theorem may hold for some functions which are not continuous. For example it holds for the following function:
$$f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0 & x = 0 \end{cases}, \qquad 0 \le x \le 1.$$

Imagine the graphs of two continuous functions f and g. If they cross each other where $x = c$ then $f(c) = g(c)$. If the graph f is higher at a and the graph of g is higher at b then they must cross, as is shown below.

Example 2.6 Let $f, g : [a, b] \to \mathbf{R}$ be continuous functions. Suppose that $f(a) < g(a)$ and $f(b) > g(b)$. Then there exists $c \in (a, b)$ such that $f(c) = g(c)$.

Proof Define $h = f - g$. Then $h(a) < 0$ and $h(b) > 0$ and h is continuous. Apply the IVT to h: there exists $c \in (a, b)$ with $h(c) = 0$, which means $f(c) = g(c)$. □

Let $f : E \to \mathbf{R}$ be a function. Any point $x \in E$ such that $f(x) = x$ is called a *fixed point* of f.

Theorem 2.3 (Fixed Point Theorem) *Suppose $g : [a, b] \to [a, b]$ is a continuous function. Then there exists $c \in [a, b]$ such that $g(c) = c$.*

Proof The notation that $g : [a, b] \to [a, b]$ implies that the range of f is contained in $[a, b]$.

Set $f(x) = g(x) - x$. Then $f(a) = g(a) - a \geq a - a = 0$, $f(b) = g(b) - b \leq b - b = 0$.

- If $f(a) = 0$ then a is the sought after point.
- If $f(b) = 0$ then b is the sought after point.
- If $f(a) > 0$ and $f(b) < 0$, apply the intermediate value theorem to f to see that there is a point $c \in (a, b)$ such that $f(c) = 0$. This means $g(c) = c$.

□

Remark This theorem has a remarkable generalisation to higher dimensions, known as Brouwer's Fixed Point Theorem. Its statement requires the notion of continuity for functions whose domain and range are of higher dimension - in this case a function from a product of intervals $[a, b]^n$ to itself. Note that $[a, b]^2$ is a square, and $[a, b]^3$ is a cube. I invite you to adapt the definition of continuity we have given, to this higher-dimensional case.

Theorem 2.4 *(Brouwer, 1912) Let $f : [a, b]^n \to [a, b]^n$ be a continuous function. Then f has a fixed point.*

The proof of Brouwer's theorem when $n > 1$ is rather harder than when $n = 1$. It uses the techniques of Algebraic Topology.

Chapter 3

Continuous Limits

3.1 Continuous Limits – Lecture 7

We wish to give a precise meaning to the statement "f approaches ℓ as x approaches c", which is denoted by

$$\lim_{x \to c} f(x) = \ell$$

The domain E of the function should include a neighbourhood of c (this neighbourhood can be very small) but does not have to contain c. Let $E = (a, b)/\{c\}$. Note that

$$(a, b)/\{c\} = (a, c) \cup (c, b).$$

We sometimes use backslash instead of the forward slash for the exclusion of c: $(a, b)/\{c\} = (a, b)\backslash\{c\}$. See the Appendix, Chapter 3.2, for the case when E is not of the form $(a, c) \cup (c, b)$.

Definition 3.1 *Let $c \in (a, b)$ and let $f : (a, b)/\{c\} \to \mathbf{R}$. Let ℓ be a real number. We say that $\lim_{x \to c} f(x)$ exists and equals ℓ, if for any $\epsilon > 0$ there exists $\delta > 0$, such that for all x satisfying*

$$0 < |x - c| < \delta \quad \text{and} \quad x \in (a, b)/\{c\}, \tag{3.1}$$

we have

$$|f(x) - \ell| < \epsilon.$$

In short we write $\lim_{x \to c} f(x) = \ell$.

Remark 3.1 1. Note the condition $|x-c| > 0$ means x is not allowed to be c. When we consider $\lim_{x\to c} f(x)$ we do not care about the value of f at c. And for that matter f does not need to be defined at c. That is why the domain of f did not have to include c.

2. In the definition, we may assume that the domain of f is a subset E of $\mathbf{R}$ containing $(c-r, c) \cup (c, c+r)$ for some $r > 0$.

Remark 3.2 If a function has a limit at c, this limit must be unique.

Proof Suppose f has two limits ℓ_1 and ℓ_2. Take $\epsilon = \frac{1}{4}|\ell_1 - \ell_2|$. If $\ell_1 \neq \ell_2$ then $\epsilon > 0$, so by definition of limit there exists $\delta > 0$ such that if $0 < |x-c| < \delta$, $|f(x) - \ell_1| < \epsilon$ and $|f(x) - \ell_2| < \epsilon$. By the triangle inequality,

$$|\ell_1 - \ell_2| \le |f(x) - \ell_1| + |f(x) - \ell_2| < 2\epsilon = \frac{1}{2}|l_1 - l_2|.$$

This cannot happen. We must have $\ell_1 = \ell_2$. $\square$

Theorem 3.1 *Let $c \in (a,b)$. Let $f : (a,b) \to \mathbf{R}$. The following are equivalent:*

1. *f is continuous at c.*
2. $\lim_{x\to c} f(x) = f(c)$.

Proof

- Assume that f is continuous at c.

  $\forall \epsilon > 0$ there is a $\delta > 0$ such that for all x with

  $$|x-c| < \delta \quad \text{and} \quad x \in (a,b), \tag{3.2}$$

  we have

  $$|f(x) - f(c)| < \epsilon.$$

  Condition (3.2) holds if condition (3.1) holds, so $\lim_{x\to c} f(x) = f(c)$.

- On the other hand suppose that $\lim_{x\to c} f(x) = f(c)$. For any $\epsilon > 0$ there exists $\delta > 0$, such that for all $x \in (a,b)$ satisfying

  $$0 < |x-c| < \delta$$

  we have

  $$|f(x) - \ell| < \epsilon.$$

If $|x-c| = 0$ then $x = c$. In this case $|f(c) - f(c)| = 0 < \epsilon$. Hence for all $x \in (a,b)$ with $|x-c| < \delta$, we have $|f(x) - \ell| < \epsilon$. Thus f is continuous at c.

$\square$

Example 3.1 Does
$$\lim_{x\to 1} \frac{x^2+3x+2}{\sin(\pi x)+2}$$
exist?

Proof Let
$$f(x) = \frac{x^2+3x+2}{\sin(\pi x)+2}.$$
Then f is continuous on all of $\mathbf{R}$, for both the numerator and the denominator are continuous on all of $\mathbf{R}$, and the numerator is never zero. Hence $\lim_{x\to 1} f(x) = f(1) = \frac{6}{2} = 3$. $\square$

Example 3.2 Does
$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$
exist? Compute its value if it does.

Proof For $x \neq 0$ and x closes to zero (e.g. for $|x| < 1/2$), the function
$$f(x) = \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$
is well defined. When $x \neq 0$,
$$f(x) = \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})} = \frac{2}{\sqrt{1+x}+\sqrt{1-x}}.$$
Since $\sqrt{x}$ is continuous at $x = 1$, (prove it by $\epsilon - \delta$ argument!), the functions $x \mapsto \sqrt{1+x}$ and $x \mapsto \sqrt{1-x}$ are both continuous at $x = 0$; as their sum is non-zero when $x = 0$, it follows, by the algebra of continuous functions, that the function
$$f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbf{R}, \qquad f(x) = \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$
is also continuous at $x = 0$. Hence
$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = f(0) = 1.$$

$\square$

Example 3.3 The statement that $\lim_{x\to 0} f(x) = 0$ is equivalent to the statement that $\lim_{x\to 0} |f(x)| = 0$.

Proof Define $g(x) = |f(x)|$. Then $|g(x) - 0| = |f(x)|$. The statement $|g(x) - 0| < \epsilon$ is the same as the statement $|f(x) - 0| < \epsilon$. □

Remark 3.3 (use of negation) The statement "it is not true that $\lim_{x\to c} f(x) = \ell$" means precisely that there exists a number $\epsilon > 0$ such that for all $\delta > 0$ there exists a $x \in (a,b)$ with $0 < |x - c| < \delta$ and $|f(x) - f(c)| \ge \epsilon$. This can occur in two ways:

1. the limit does not exist, or
2. the limit exists, but differs from ℓ.

In the second case, but not in the first case, we write

$$\lim_{x\to c} f(x) \ne \ell.$$

In the first case it would be wrong to write this since it suggests that the limit exists.

Theorem 3.2 *Let $c \in (a,b)$ and $f : (a,b)/\{c\} \to \mathbf{R}$. The following are equivalent.*

1. $\lim_{x\to c} f(x) = \ell$
2. *For every sequence $x_n \in (a,b)/\{c\}$ with $\lim_{n\to\infty} x_n = c$ we have*

$$\lim_{n\to\infty} f(x_n) = \ell.$$

Proof

- Step 1. If $\lim_{x\to c} f(x) = \ell$, define

$$g(x) = \begin{cases} f(x), & x \ne c \\ \ell, & x = c \end{cases}$$

  Then $\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \ell = g(c)$. This means that g is continuous at c. If x_n is a sequence with $x_n \in (a,b)/\{c\}$ and $\lim_{n\to\infty} x_n = c$, then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = \ell$$

  by sequential continuity of g at c. We have shown that statement 1 implies statement 2.

- Step 2. Suppose that $\lim_{x\to c} f(x) = \ell$ does not hold.

  There is a number $\epsilon > 0$ such that for all $\delta > 0$, there is a number $x \in (a,b)/\{c\}$ with $0 < |x-c| < \delta$, but $|f(x) - \ell| \geq \epsilon$.

  Taking $\delta = \frac{1}{n}$ we obtain a sequence $x_n \in (a,b)/\{c\}$ with $|x_n - c| < \frac{1}{n}$ with

$$|f(x_n) - \ell| \geq \epsilon$$

  for all n. Thus the statement $\lim_{n\to\infty} f(x_n) = \ell$ cannot hold. But x_n is a sequence with $\lim_{n\to\infty} x_n = c$. Hence statement 2 fails.

□

Example 3.4 Let $f : \mathbf{R}/\{0\} \to \mathbf{R}$, $f(x) = \sin(\frac{1}{x})$. Claim:

$$\lim_{x\to 0} \sin(\frac{1}{x})$$

does not exist (DNE).

Proof Take two sequences of points,

$$\begin{aligned} x_n &= \frac{1}{\frac{\pi}{2} + 2n\pi} \\ z_n &= \frac{1}{-\frac{\pi}{2} + 2n\pi}. \end{aligned}$$

Both sequences tend to 0 as $n \to \infty$. But $f(x_n) = 1$ and $f(x_n) = -1$ so $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(z_n)$. By the sequential formulation for limits, Theorem 3.2, $\lim_{x\to 0} f(x)$ cannot exist. □

Example 3.5 Let $f : \mathbf{R}/\{0\} \to \mathbf{R}$, $f(x) = \frac{1}{|x|}$. Claim: there is no number ℓ s.t. $\lim_{x\to 0} f(x) = \ell$.

Proof Suppose that there exists ℓ such that $\lim_{x\to 0} f(x) = \ell$. Take $x_n = \frac{1}{n}$. Then $f(x_n) = n$ and $\lim_{n\to\infty} f(x_n)$ does not converges to any finite number! This contradicts that $\lim_{n\to\infty} f(x_n) = \ell$. □

Example 3.6 Denote by $[x]$ the integer part of x. Let $f(x) = [x]$. Show that $\lim_{x\to 1} f(x)$ does not exist.

Proof We only need to consider the function f near 1. Let us consider f on $(0,2)$. Then

$$f(x) = \begin{cases} 1, & \text{if } 1 \le x < 2 \\ 0, & \text{if } 0 \le x < 1 \end{cases}.$$

Let us take a sequence $x_n = 1 + \frac{1}{n}$ converging to 1 from the right and a sequence $y_n = 1 - \frac{1}{n}$ converging to 1 from the left . Then

$$\lim_{n\to\infty} x_n = 1, \qquad \lim_{n\to\infty} y_n = 1.$$

Since $f(x_n) = 1$ and $f(y_n) = 0$ for all n, the two sequences $f(x_n)$ and $f(y_n)$ have different limits and hence $\lim_{x\to 1} f(x)$ does not exist. □

The following follows from the sandwich theorem for sequential limits.

Proposition 3.3 (Sandwich Theorem/Squeezing Theorem) *Let $c \in (a,b)$ and $f, g, h : (a,b)/\{c\} \to \mathbf{R}$. If $h(x) \le f(x) \le g(x)$ on (a,b), and*

$$\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = \ell$$

then

$$\lim_{x\to c} f(x) = \ell.$$

Proof Let $x_n \in (a,b)/\{c\}$ be a sequence converging to c. Then by Theorem 3.2, $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} h(x_n) = \ell$. Since $h(x_n) \le f(x_n) \le g(x_n)$,

$$\lim_{x\to c} f(x_n) = \ell.$$

By Theorem 3.2 again, we see that $\lim_{x\to c} f(x) = \ell$. □

Example 3.7 Prove that

$$\lim_{x\to 0} x \sin(\frac{1}{x}) = 0.$$

Proof Note that $0 \le |x \sin(\frac{1}{x})| \le |x|$. Now $\lim_{x\to 0} |x| = 0$ by the the continuity of the function $f(x) = |x|$. By the Sandwich theorem the conclusion $\lim_{x\to 0} x \sin(\frac{1}{x}) = 0$ follows. □

Example 3.8 Claim:

$$f(x) = \begin{cases} x \sin(1/x) & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is continuous at 0.

This claim follows from example 3.7, $\lim_{x\to 0} f(x) = 0 = f(0)$.

The following eample will be re-visited in Example 12.2 (an application of L'Hôpital's rule).

Example 3.9 (Important limit to memorise)

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Proof Recall if $0<x<\frac{\pi}{2}$, then $\sin x\le x\le \tan x$ and

$$\frac{\sin x}{\tan x}\le \frac{\sin x}{x}\le 1.$$

$$\cos x\le \frac{\sin x}{x}\le 1 \tag{3.3}$$

The relation (3.3) also holds if $-\frac{\pi}{2}<x<0$: Letting $y=-x$ then $0<y<\frac{\pi}{2}$. All three terms in (3.3) are even functions: $\frac{\sin x}{x}=\frac{\sin y}{y}$ and $\cos y=\cos(-x)$.

Since

$$\lim_{x\to 0}\cos x=1$$

by the Sandwich Theorem and (3.3), $\lim_{x\to 0}\frac{\sin x}{x}=1$.

□

From the algebra of continuity and the continuity of composites of continuous functions we deduce the following for continuous limits, with the help of Theorem 3.2.

Proposition 3.4 (Algebra of limits) *Let $c\in(a,b)$ and $f,g(a,b)/\{c\}$. Suppose that*

$$\lim_{x\to c}f(x)=\ell_1,\qquad \lim_{x\to c}g(x)=\ell_2.$$

Then

1.
$$\lim_{x\to c}(f+g)(x)=\lim_{x\to c}f(x)+\lim_{x\to c}g(x)=\ell_1+\ell_2.$$

2.
$$\lim_{x\to c}(fg)(x)=\lim_{x\to xc}f(x)\lim_{x\to c}g(x)=\ell_1\ell_2.$$

3. *If $\ell_2\ne 0$,*
$$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)}=\frac{\ell_1}{\ell_2}.$$

Proposition 3.5 (Limit of composition) *Let $c \in (a,b)$ and $f : (a,b)/\{c\} \to \mathbf{R}$. Suppose that*

$$\lim_{x \to c} f(x) = l.$$

Suppose that the range of f is a subset of (a_1, b_1). Suppose that

$$g : (a_1, b_1)/\{l\} \to \mathbf{R}$$

and $\lim_{x \to l} g(x) = K$ Then

$$\lim_{x \to c} g(f(x)) = K.$$

The following example will be re-visited in Example 12.4 (an application of L'Hôpital's rule).

Example 3.10 (Important limit to memorise) Show that

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$

Proof.

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos x}{x} &= \lim_{x \to 0} \frac{2 \sin^2(\frac{x}{2})}{x} \\
&= \lim_{x \to 0} \sin(\frac{x}{2}) \frac{\sin(\frac{x}{2})}{\frac{x}{2}} \\
&= \lim_{x \to 0} \sin(\frac{x}{2}) \times \lim_{x \to 0} \frac{\sin(\frac{x}{2})}{\frac{x}{2}} \\
&= 0 \times 1 = 0.
\end{aligned}$$

The following lemma should be compared to Lemma 1.4. The existence of the $\lim_{x \to c} f(x)$ and its value depend only on the behaviour of f near to c. It is in this sense we say that 'limit' is a local property. Let $B_r(c) = (c - r, c + r)$ and $\mathring{B}_r(c) = (c - r, c + r)/\{c\}$. If a property holds on $B_r(c)$ for some $r > 0$, we often say that the property holds "close to c".

Lemma 3.6 *Let $g : (a,b)/\{c\} \to \mathbf{R}$, and $c \in (a,b)$. Suppose that*

$$\lim_{x \to c} g(x) = \ell.$$

1. *If $\ell > 0$, then $g(x) > 0$ on $\mathring{B}_r(c)$ for some $r > 0$.*
2. *If $\ell < 0$, then $g(x) < 0$ on $\mathring{B}_r(c)$ for some $r > 0$.*

In both cases $g(x) \neq 0$ *for* $x \in \mathring{B}_r(c)$.

Proof Define

$$\tilde{g}(x) = \begin{cases} g(x), & x \neq c \\ \ell, & x = c. \end{cases}$$

Then $\tilde{g}$ is continuous at c and so by Lemma 1.4, $\tilde{g}$ is not zero on $(c-r, c+r)$ for some r. The conclusion for g follows. $\square$

Proof [Direct Proof] Suppose that $l > 0$. Let $\epsilon = \frac{l}{2} > 0$. There exists $r > 0$ such that if $x \in (\mathring{B}_r(c)$ and $x \in (a,b)$, then.

$$g(x) > \ell - \epsilon = \frac{l}{2} > 0.$$

We choose $r < \min(b-c, c-a)$ so that $(c-r, c+r) \subset (a,b)$.

If $\ell < 0$ let $\epsilon = -\frac{\ell}{2} > 0$, then there exists $r > 0$, with $r < \min(b-c, c-a)$, such that on $\mathring{B}_r(c)$,

$$g(x) < l + \epsilon = l - \frac{l}{2} = \frac{l}{2} < 0.$$

$\square$

3.2 Accumulation points and Limits*

This section is not delivered in the lectures. Recall that a point x_0 is called an *accumulation point* (or *limit point*) of a set E if for any $\delta > 0$, there is a point $x \in A$ such that $0 < |x - x_0| < \delta$.

Example 3.11

- If $E = (1,2) \cup [3,4)$ its set of accumulation point is $[1,2] \cup [3,4]$.
- $\{1, 2, \dots, \}$ has no accumulation points.
- Let $E = \{a_1, a_2, \dots\}$. If a is the limit of a sub-sequence of $\{a_n\}$, then a is an accumulation point of E.
- Every real number is an accumulation point of $\mathbf{Q}$.
- Every real number is an accumulation point of $\mathbf{R} - \mathbf{Q}$.

Definition 3.2 *Let c be an accumulation point of E. A function $f : E \to \mathbf{R}$ has a limit ℓ as x approaches c, where $\ell \in \mathbf{R}$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in E$ satisfying $0 < |x - c| < \delta$, $|f(x) - \ell| < \epsilon$.*

In this case we write

$$\lim_{x \to c} f(x) = \ell.$$

Example 3.12 Why do we need c to be an accumulation point of E? Take $E = \{\frac{1}{n}\}$, define $f(\frac{1}{n}) = n$. Let $c = \frac{1}{2}$, not an accumulation point. Then any number l could be the limit of $f(x)$ as $x \to c = \frac{1}{2}$. In fact if we take $\delta = \frac{1}{6}$, then no element of E satisfies $0 < |x - \frac{1}{2}| < \delta$. Hence if $0 < |x - \frac{1}{2}| < \delta = \frac{1}{6}$ and $x \in E$, $|f(x) - l| < \epsilon$ for any number ℓ! The concept of limit in this case is absurd.

Proposition 3.7 *If $\lim_{x \in c} f(x)$ exists, then the limit must be unique.*

Proof If f has two limits u and v, take $\epsilon = \frac{1}{4}|u - v|$. Then there is $\delta > 0$ such that if $0 < |x - c| < \delta$, $|u - v| \le |f(x) - u| + |f(x) - v| < \epsilon$. Such x exists as c is an accumulation point of A, giving a contradiction. $\square$

Proposition 3.8 *The following statements are equivalent:*

1. *f has a limit at c*
2. *for any $\epsilon > 0$ there is $\delta > 0$, such that for all $x, y \in E$ satisfying $0 < |x - c| < \delta$ and $0 < |y - c| < \delta$ we have $|f(x) - f(y)| < \epsilon$.*

Proof

1. Suppose that Statement 1 holds. Let $\ell = \lim_{x \to c} f(x)$. Then for any $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in E$ satisfying $0 < |x - c| < \delta$ and $0 < |y - c| < \delta$,

   $$|f(x) - \ell| < \frac{\epsilon}{2}, \qquad |f(y) - \ell| < \frac{\epsilon}{2}.$$

   By the triangle inequality,

   $$|f(x) - f(y)| \le |f(x) - \ell| + |f(x) - \ell| < \epsilon.$$

2. We assume that Statement 2 holds and prove that f has a limit at c by contradiction.

   Suppose $\lim_{x \to c} f(x) = \ell$ for some number ℓ. Then for any sequence $x_n \to c$, $\lim_{n \to \infty} f(x_n) = \ell$.

Statement 2 fails means that: there is $\epsilon_0 > 0$ such that for all $\delta > 0$ there are $x, y \in E$ such that $0 < |x - c| < \delta$, $0 < |y - c| < \delta$ and $|f(x) - f(y)| > \epsilon_0$. Take $\delta = \frac{1}{n}$, we obtain a sequence $\{x_n\}$, $\{y_n\}$ such that $x_n, y_n \in E$, $0 < |x_n - c| < \frac{1}{n}$, $0 < |y_n - c| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \epsilon_0$.

Now $\lim_{n\to\infty} f(x_n) = l$ and $\lim_{n\to\infty} f(y_n) = l$ and so

$$|f(x_n) - f(y_n)| \le |f(x_n - l| + |f(y_n) - l|$$

can be as small as we like provided that we take n large. This contradicts $|f(x_n) - f(y_n)| > \epsilon_0$. Hence Statement 2 fails implies that Statement 1 fails.

□

Definition 3.3 *Consider $f : E \to \mathbf{R}$ and c an accumulation point of E.*

1. *We say $\lim_{x\to c} f(x) = \infty$, if $\forall M > 0$, $\exists \delta > 0$, such that $f(x) > M$ whenever $x \in E$ satisfies $0 < |x - c| < \delta$.*

2. *We say $\lim_{x\to c} f(x) = -\infty$, if $\forall M > 0$, $\exists \delta > 0$, such that $f(x) < -M$ whenever $x \in E$ satisfies $0 < |x - c| < \delta$.*

3.3 One Sided Limits – Lecture 8

How do we define "the limit of f as x approaches c from the right " or "the limit of f as x approaches c from the left "?

Definition 3.4 *Let $c \in (a, b)$. A function $f : (a, b)/\{c\} \to \mathbf{R}$ has a left limit ℓ at c if for any $\epsilon > 0$ there exists $\delta > 0$, such that for all*

$$x \in (a, c) \quad \textit{with} \quad c - \delta < x < c \tag{3.4}$$

we have

$$|f(x) - \ell| < \epsilon.$$

In this case we write

$$\lim_{x\to c-} f(x) = \ell.$$

Right limits are defined in a similar way:

Definition 3.5 *Let $c \in (a,b)$. We say that $\lim_{x\to c+} f(x) = \ell$ if for any $\epsilon > 0$ there exists $\delta > 0$, such that for all $x \in (c,b)$ with $c < x < c+\delta$ we have $|f(x) - \ell| < \epsilon$.*

Remark 3.4

- (3.4) is equivalent to
$$x \in (a,c), \qquad \& \quad 0 < |x-c| < \delta.$$
- $\lim_{x\to c} f(x) = \ell$ if and only if both $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ exist and are equal to ℓ.

Definition 3.6

1. *A function $f : (a,c) \to \mathbf{R}$ is said to be left continuous at c if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $c - \delta < x < c$ then*
$$|f(x) - f(c)| < \epsilon.$$
2. *A function $f : (c,b) \to \mathbf{R}$ is said to be right continuous at c if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $c < x < c + \delta$*
$$|f(x) - f(c)| < \epsilon.$$

Theorem 3.9 *Let $f : (a,b) \to \mathbf{R}$ and for $c \in (a,b)$. The following are equivalent:*

(a) f is continuous at c

(b) f is right continuous at c and f is left continuous at c.

(c) Both $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ exist and are equal to $f(c)$.

(d) $\lim_{x\to c} f(x) = f(c)$

Example 3.13 Denote by $[x]$ the integer part of x. Let $f(x) = [x]$. Show that for $k \in \mathbf{Z}$, $\lim_{x\to k+} f(x)$ and $\lim_{x\to k-} f(x)$ exist. Show that $\lim_{x\to k} f(x)$ does not exist. Show that f is discontinuous at all points $k \in \mathbf{Z}$.

Proof Let $c = k$, we consider f near k, say on the open interval $(k-1, k+1)$. Note that
$$f(x) = \begin{cases} k, & \text{if } k \le x < k+1 \\ k-1, & \text{if } k-1 \le x < k \end{cases}.$$
It follows that
$$\begin{aligned} \lim_{x\to k+} f(x) &= \lim_{x\to 1+} k = k \\ \lim_{x\to k-} f(x) &= \lim_{x\to 1-} (k-1) = k-1. \end{aligned}$$

Since the left limit does not agree with the right limit, $\lim_{x \to k} f(x)$ does not exist. By the limit formulation of continuity, ('A function continuous at k must have a limit as x approaches k'), f is not continuous at k. $\square$

Example 3.14 For which number a is f defined below continuous at $x = 0$?

$$f(x) = \begin{cases} a, & x \neq 0 \\ 2, & x = 0 \end{cases} .$$

Since $\lim_{x \to 0} f(x) = 2$, f is continuous if and only if $a = 2$.

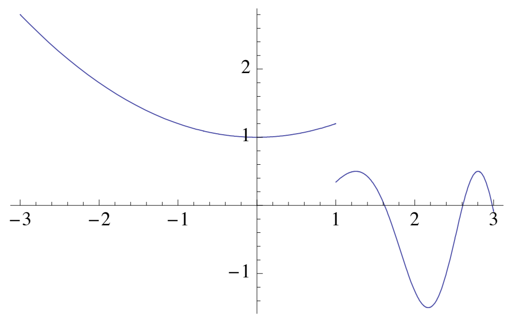
Let $c \in (a, b)$. If $f(a, b) \to \mathbf{R}$ is continuous everywhere except at c, how could this happen? Is it left continuous? Is it right continuous? Does it have a left limit? Does it have a right limit? Give examples.

We show graphs of a few functions. They are graphed using Mathematica. They are, in the order they appear,

$$f(x) = \begin{cases} \frac{1}{5}x^2 + 1, & \text{if } -3 < x < 1 \\ \sin(x^2) - 0.5, & \text{if } 1 \geq x < 3 \end{cases} .$$

$f(x) = \frac{1}{|x|}$, $f(x) = \frac{1}{x}$, $f(x) = \sin(1/x), x \in [0, 1.5]$, $f(x) = \sin(1/x), x \in [0, 0.01]$ and $f(x) = \frac{x}{1+x}$.

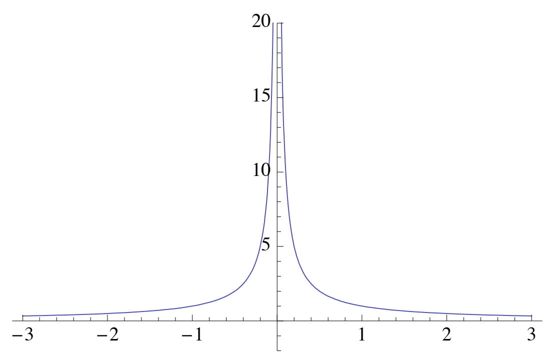
In[4]:= `Plot[Piecewise[{{(1 / 5) x^2 + 1, x < 1}, {Sin [x^2] - 0.5, x > 1}}], {x, -3, 3}]`



In[10]:=

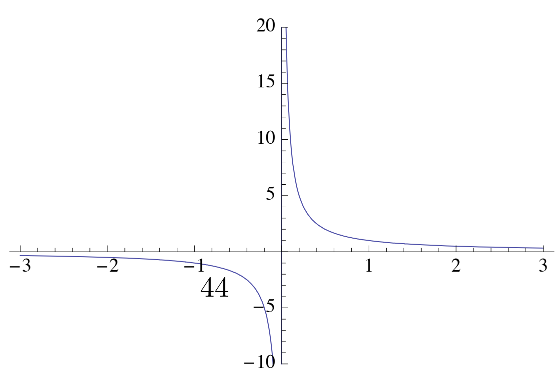
`Plot[{1 / Abs[x]}, {x, -3, 3}, PlotRange → {-2, 20}]`

Out[10]=



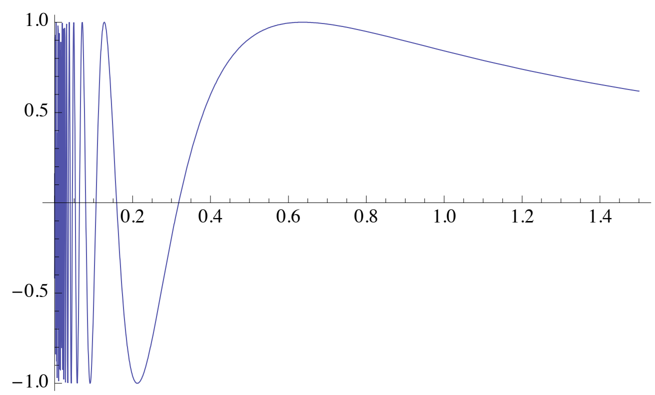
`Plot[{1 / x}, {x, -3, 3}, PlotRange → {-20, 20}]`

Out[9]=



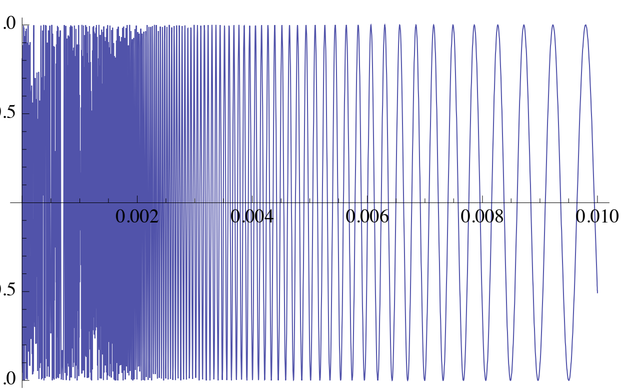
In[22]:= `Plot[Sin[1 / x], {x, 0, 1.5}]`

Out[22]=



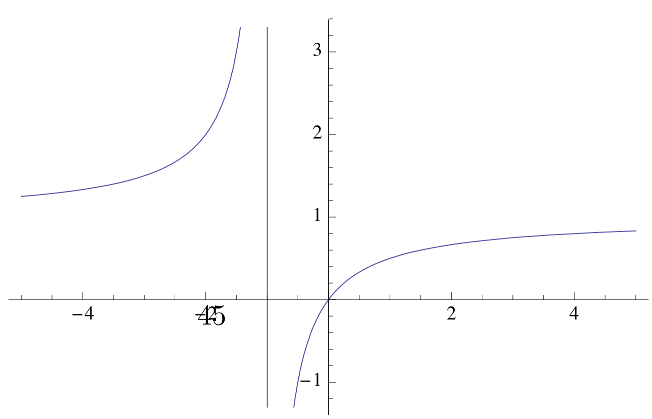
In[21]:= `Plot[Sin[1 / x], {x, 0, 0.01}]`

Out[21]=



In[23]:= `Plot[x / (1 + x), {x, -5, 5}]`

Out[23]=



3.4 Limits to ∞

We make sense of the statement that 'f approaches infinity' and 'x approaches c'. What do we mean by

$$\lim_{x \to c} f(x) = \infty?$$

Caution: ∞ is not a real number. We must formulate what it means to approach it.

Definition 3.7 *Let $c \in (a,b)$ and $f : (a,b)/\{c\} \to \mathbf{R}$. We say that*

$$\lim_{x \to c} f(x) = \infty,$$

if $\forall M > 0$, $\exists \delta > 0$, such that (s.t.) for all $x \in (a,b)$ with $0 < |x-c| < \delta$ we have

$$f(x) > M.$$

Example 3.15 Show that

$$\lim_{x \to 0} \frac{1}{|x|} = \infty$$

Proof Let $M > 0$. Define $\delta = \frac{1}{M}$. Then if $0 < |x-0| < \delta$, we have $f(x) = \frac{1}{|x|} > \frac{1}{\delta} = M$. $\square$

Definition 3.8 *Let $c \in (a,b)$ and $f : (a,b)/\{c\} \to \mathbf{R}$. We say that*

$$\lim_{x \to c} f(x) = -\infty,$$

if $\forall M > 0$, $\exists \delta > 0$, such that

$$f(x) < -M$$

for all $x \in (a,b)$ with $0 < |x-c| < \delta$.

One sided limits can be similarly defined. For example,

Definition 3.9 *Let $f : (c,b) \to \mathbf{R}$. We say that*

$$\lim_{x \to c+} f(x) = \infty,$$

if $\forall M > 0$, $\exists \delta > 0$, s.t. $f(x) > M$ for all $x \in (c,b) \cap (c, c+\delta)$.

Definition 3.10 *Let $f : (a, c) \to \mathbf{R}$. We say that*

$$\lim_{x \to c-} f(x) = -\infty,$$

if $\forall M > 0$, $\exists \delta > 0$, such that $f(x) < -M$ for all $x \in (a, c)$ with $c - \delta < x < c$.

Exercise 3.16 Show that $\lim_{x \to 0-} \frac{1}{\sin x} = -\infty$.

Proof Let $M > 0$. Since $\lim_{x \to 0} \sin x = 0$, there is $\delta > 0$ such that if $0 < |x| < \delta$, then $|\sin x| < \frac{1}{M}$.

Since we are interested in the left limit, we may consider $x \in (-\frac{\pi}{2}, 0)$. In this case $|\sin x| = -\sin x$. So we have $-\sin x < \frac{1}{M}$ which is the same as

$$\sin x > -\frac{1}{M}.$$

In conclusion $\lim_{x \to 0-} \frac{1}{\sin x} = -\infty$. $\square$

3.5 Limits at ∞

What do we mean by

$$\lim_{x \to \infty} f(x) = \ell, \quad \text{or} \quad \lim_{x \to \infty} f(x) = \infty?$$

Definition 3.11 1. *Consider $f : (a, \infty) \to \mathbf{R}$. We say that*

$$\lim_{x \to \infty} f(x) = \ell,$$

if for any $\epsilon > 0$ there is an M such that if $x > M$ we have $|f(x) - \ell| < \epsilon$.

2. *Consider $f : (-\infty, b) \to \mathbf{R}$. We say that*

$$\lim_{x \to -\infty} f(x) = \ell,$$

if for any $\epsilon > 0$ there is an $M > 0$ such that for all $x < -M$ we have $|f(x) - \ell| < \epsilon$.

Definition 3.12 *Consider $f : (a, \infty) \to \mathbf{R}$. We say*

$$\lim_{x \to \infty} f(x) = \infty,$$

if $\forall M > 0$, $\exists$ a number $X > 0$, such that

$$f(x) > M, \text{ for all } x > X.$$

Example 3.17 Show that $\lim_{x\to+\infty}(x^2+1) = +\infty$.

Proof For any $M > 0$, we look for x with the property that

$$x^2 + 1 > M.$$

Take $A = \sqrt{M}$ then if $x > A$, $x^2+1 > M+1$. Hence $\lim_{x\to+\infty}(x^2+1) = +\infty$. $\square$

Remark 3.5 In all cases,

1. There is a unique limit if it exists.
2. There is a sequential formulation. For example, $\lim_{x\to\infty} f(x) = \ell$ if and only if $\lim_{n\to\infty} f(x_n) = \ell$ for all sequences $\{x_n\} \subset E$ with $\lim_{n\to\infty} x_n = \infty$. Here E is the domain of f.
3. Algebra of limits hold.
4. The Sandwich Theorem holds.

Exercise 3.18 Formulate:

$$\lim_{x\to\infty} f(x) = -\infty; \quad \lim_{x\to-\infty} f(x) = \infty \quad \lim_{x\to-\infty} f(x) = -\infty.$$

Exercise 3.19 Let $f(x) = \frac{x}{1+x}$. It is defined on $\mathbf{R}/\{-1\}$.

1. Show that the one sided limits at $c = -1$ are infinite:
$$\lim_{x\to-1+} \frac{x}{1+x} = -\infty, \qquad \lim_{x\to-1-} \frac{x}{1+x} = \infty.$$
2. Show that the limit at infinity exists:
$$\lim_{x\to\infty} \frac{x}{1+x} = 1, \qquad \lim_{x\to-\infty} \frac{x}{1+x} = 1.$$
3. Plot $y = 1$ and $x = -1$ in dotted lines. Plot on the same graph $(0, f(0))$, $(2, f(2))$, $5, f(5)$, and $(8, f(8)$.
4. Plot the graph of $y = f(x)$ on the right hand side of $x > -1$.
5. Plot the points on the graph of $y = f(x)$ corresponds to the points $x = -1.1, -1.01, -2, -5$.

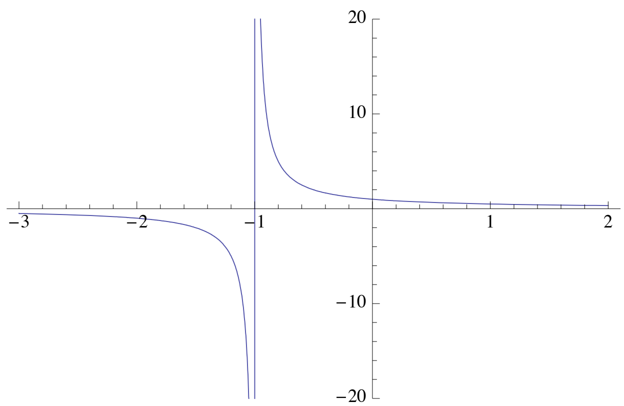
6. Now complete the graph of $y = f(x)$ to the left of $f = -1$. (see the second appended graph).

7. N.B. $f(x) = 1 - \frac{1}{1+x}$. Its graph can be obtained from shifting and rotating the graph of $y = \frac{1}{x}$.

Exercise 3.20 Show that $\lim_{x\to c} f(x) = +\infty$ implies that $\lim_{x\to c} \frac{1}{f(x)} = 0$. Does the converse hold? Give examples to illustrate your point.

In[43]:= `Plot[{1 / (1 + x)}, {x, -3, 2}, PlotRange → {-20, 20}]`

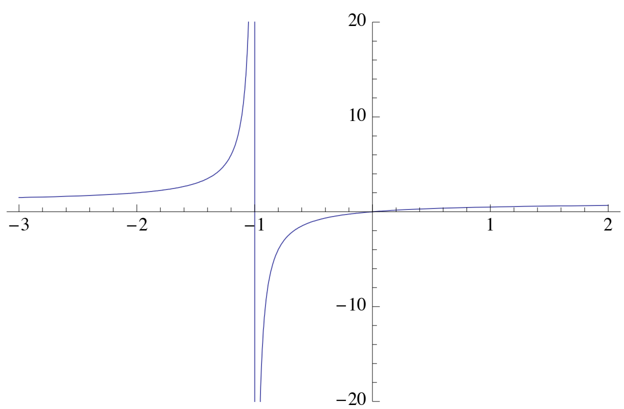
Out[43]=



In[45]:=

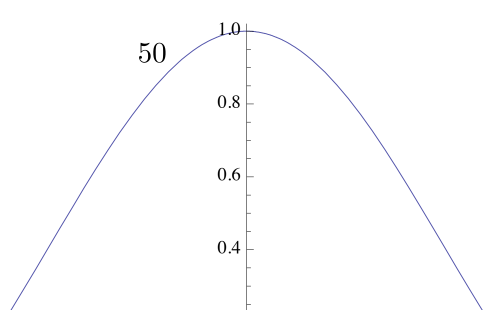
`Plot[{x / (1 + x)}, {x, -3, 2}, PlotRange → {-20, 20}]`

Out[45]=

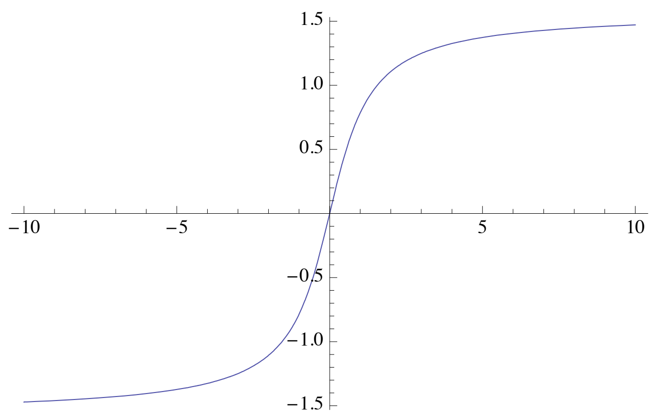


In[60]:= `Plot[Sin[x] / x, {x, -Pi, Pi}]`

Out[60]=

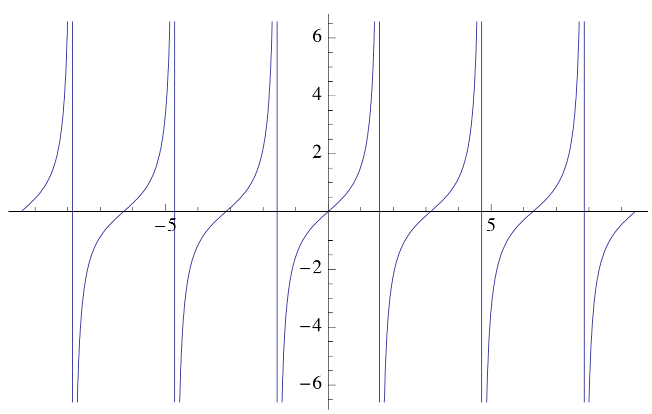


In[57]:=
```
Plot[ArcTan[x], {x, -10, 10}]
```



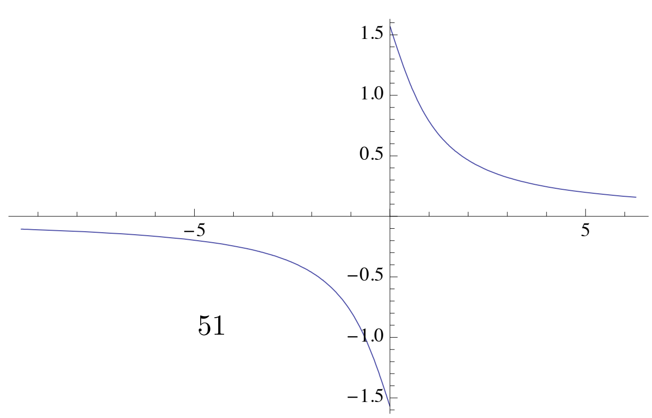
In[58]:=
```
Plot[Tan[x], {x, -3 Pi, 3 Pi}]
```

Out[58]=



In[64]:=
```
Plot[ArcCot[x], {x, -3 Pi, 2 Pi}]
```

Out[64]=



Chapter 4

The Extreme Value Theorem

By now we understood quite well what is a continuous function. Let us look at the landscape drawn by a continuous function. There are peaks and valleys. Is there a highest peak or a lowest valley?

4.1 Bounded Functions

Consider the range of $f : E \to \mathbf{R}$:

$$A = \{f(x)|x \in E\}.$$

Is A bounded from above and from below? If so it has a greatest lower bound m_0 and a least upper bound M_0. And

$$m_0 \le f(x) \le M_0, \qquad \forall x \in E.$$

Do m_0 and M_0 belong to A? That they belong to A means that there exist $\underline{x} \in E$ and $\bar{x} \in E$ such that $f(\underline{x}) = m_0$ and $f(\bar{x}) = M_0$.

Definition 4.1 *We say that $f : E \to \mathbf{R}$ is bounded above if there is a number M such that $f(x) \le M$ for all $x \in E$. We say that f attains its maximum if there is a number $c \in E$ such that $f(x) \le f(c)$.*

That a function f is bounded above means that its range $f(E)$ is bounded above.

Definition 4.2 *We say that $f : E \to \mathbf{R}$ is bounded below if there is a number m such that $f(x) \ge m$ for all x in the domain of f. We say that f attains its minimum if there is a number $c \in E$ such that $f(x) \ge f(c)$.*

That a function f is bounded below means that its range $f(E)$ is bounded below.

Definition 4.3 *A function which is bounded above and below is* **bounded***.*

4.2 The Bolzano-Weierstrass Theorem

This theorem was introduced and proved in Analysis I.

Lemma 4.1 (Bolzano-Weierstrass Theorem) *A bounded sequence has at least one convergent sub-sequence.*

4.3 The Extreme Value Theorem – Lecture 9

Theorem 4.2 (The Extreme Value theorem) *Let $f : [a,b] \to \mathbf{R}$ be a continuous function. Then*

1. *f is bounded above and below, i.e. there exist numbers m and M such that*
$$m \le f(x) \le M, \qquad \forall\, x \in [a,b].$$

2. *There exist $\underline{x}, \overline{x} \in [a,b]$ such that*
$$f(\underline{x}) \le f(x) \le f(\overline{x}), \qquad a \le x \le b.$$
So
$$f(\underline{x}) = \inf\{f(x) | x \in E\}, \qquad f(\overline{x}) = \sup\{f(x) | x \in E\}.$$

Proof

1. We show that f is bounded above. Suppose not. Then for any $M > 0$ there is a point $x \in [a,b]$ such that $f(x) > M$. In particular, for every n there is a point $x_n \in [a,b]$ such that $f(x_n) \ge n$. The sequence $(x_n)_{n\in\mathbf{N}}$ is bounded, so there is a convergent subsequence x_{n_k}; let us denote its limit by x_0. Note that $x_0 \in [a,b]$. By sequential continuity of f at x_0,
$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$$
But $f(x_{n_k}) \ge n_k \ge k$ and so
$$\lim_{k\to\infty} f(x_{n_k}) = \infty.$$

This gives a contradiction. The contradiction originated in the supposition that f is not bounded above. We conclude that f *must be* bounded from above.

2. Next we show that f attains its maximum value. Let

$$A = \{f(x)|x \in [a,b]\}.$$

Since A is not empty and bounded above, it has a least upper bound. Let

$$M_0 = \sup A.$$

By definition of supremum, if $\epsilon > 0$ then $M - \epsilon$ cannot be an upper bound of A, so there is a point $f(x) \in A$ such that $M_0 - \epsilon \le f(x) \le M_0$. Takinge $\epsilon = \frac{1}{n}$ we obtain a sequence $x_n \in [a,b]$ such that

$$M_0 - \frac{1}{n} \le f(x_n) \le M_0.$$

By the Sandwich theorem,

$$\lim_{n\to\infty} f(x_n) = M_0.$$

Since $a \le x_n \le b$, it has a convergent subsequence x_{n_k} with limit $\bar{x} \in [a,b]$. By sequential continuity of f at x_0,

$$f(\bar{x}) = \lim_{k\to\infty} f(x_{n_k}) = M_0.$$

3. Let $g(x) = -f(x)$. By step 1, g is bounded above and so f is bounded below. Note that

$$\sup\{g(x)|x \in [a,b]\} = -\inf\{f(x)|x \in [a,b]\}.$$

By the conclusion of step 2, there is a point $\underline{x}$ such that $g(\underline{x}) = \sup\{g(x)|x \in [a,b]\}$. This means that $f(\underline{x}) = \inf\{f(x)|x \in [a,b]\}$.

$\square$

Remark 4.1 By the extreme value theorem, if $f : [a,b] \to \mathbf{R}$ is a continuous function then the range of f is the closed interval $[f(\underline{x}), f(\bar{x})]$. By the Intermediate Value Theorem (IVT), $f : [a,b] \to [f(\underline{x}), f(\bar{x})]$ is surjective.

Discussion on the assumptions:

Example 4.1

- Let $f : (0, 1] \to \mathbf{R}$, $f(x) = \frac{1}{x}$. Note that f is not bounded. The condition that the domain of f be a closed interval is violated.
- Let $g : [1, 2) \to \mathbf{R}$, $g(x) = x$. It is bounded above and below. But the value

  $$\sup\{g(x)|x \in [1, 2)\} = 2$$

  is not attained on $[1, 2)$. Again, the condition "closed interval" is violated.

Example 4.2 Let $f : [0, \pi] \cap \mathbf{Q}$, $f(x) = x$. Let $A = \{f(x)|x \in [0, \pi] \cap \mathbf{Q}\}$. Then $\sup A = \pi$. But π is not attained on $[0, \pi] \cap \mathbf{Q}$. The condition which is violated here is that the domain of f must be an interval.

Example 4.3 Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x$. Then f is not bounded. The condition "bounded interval" is violated.

Example 4.4 Let $f : [-1, 1] \to \mathbf{R}$,

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

Then f is not bounded. The condition "f is continuous" is violated.

Chapter 5

The Inverse Function Theorem for continuous functions

In this chapter we answer the following question: If f has an inverse and f is continuous, is f^{-1} continuous?

5.1 The Inverse of a Function

Definition 5.1 *Let E and B be subsets of $\mathbf{R}$.*

1. *We say $f : E \to B$ is injective $f(x) \neq f(y)$ whenever $x \neq y$. Here $x, y \in E$.*
2. *We say $f : E \to B$ is surjective, if for any $y \in B$ there is $x \in E$ such that $f(x) = y$.*
3. *We say f is bijective if it is surjective and injective.*

A bijection $f : E \to B$ has an inverse $f^{-1} : B \to E$:

Definition 5.2 *If $f : E \to B$ is a bijection, we define $f^{-1} : B \to E$ by*

$$f^{-1}(y) = x \quad \textit{if} \quad f(x) = y.$$

N.B.

1) If $f : E \to B$ is a bijection, then so is its inverse function $f^{-1} : B \to E$.

2) If $f : E \to \mathbf{R}$ is injective it is a bijection from its domain E to its range B. So it has an inverse: $f^{-1} : B \to E$.

3) If f^{-1} is the inverse of f, then f is the inverse of f^{-1}.

5.2 Monotone Functions

The simplest injective function is an increasing function, or a decreasing function. They are called monotone functions.

Definition 5.3 *Consider the function $f : E \to \mathbf{R}$.*

1. *We say f is increasing, if $f(x) < f(y)$ whenever $x < y$ and $x, y \in E$.*

2. *We say f is decreasing, if $f(x) > f(y)$ whenever $x < y$ and $x, y \in E$.*

3. *It is monotone or 'strictly monotone' if it is either increasing or decreasing.*

Compare this with the following definition:

Definition 5.4 *Consider the function $f : E \to \mathbf{R}$.*

1. *We say f is non-decreasing if for any pair of points x, y with $x < y$, we have $f(x) \le f(y)$.*

2. *We say f is non-increasing if for any pair of points x, y with $x < y$, we have $f(x) \le f(y)$.*

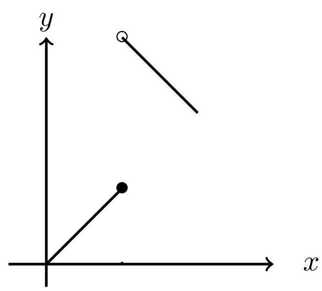
Some authors use the term 'strictly increasing' for 'increasing', and use the term 'increasing' where we use 'non-decreasing'. *We will always use 'increasing' and 'decreasing' in the strict sense defined in 5.3.*

If $f : [a, b] \to Range(f)$ is an increasing function, its inverse f^{-1} is also increasing. If f is decreasing, f^{-1} is also decreasing (Prove this!).

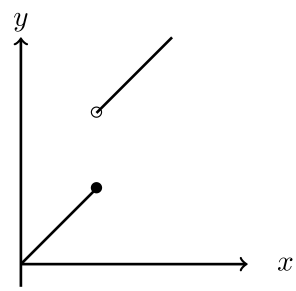
5.3 'Continuous Injective' and 'Monotone Surjective' Functions – Lecture 10

Increasing functions and decreasing functions are injective. Are there any other injective functions?

Yes: the function indicated in Graph A below is injective but not montone. Note that f is not continuous. Surprisingly, if f is continuous and injective then it must be monotone.



Graph A



Graph B

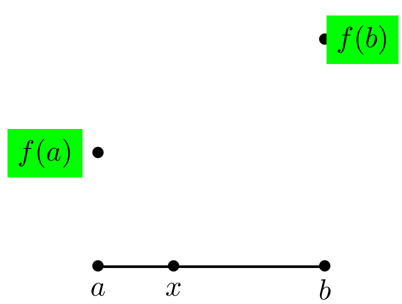
If $f : [a, b]$ is increasing, is $f : [a, b] \to [f(a), f(b)]$ surjective? Is it necessarily continuous? The answer to both questions is No, see Graph B. Again, continuity plays a role here. We show below that for an increasing function, 'being surjective' is equivalent to 'being continuous'!

Theorem 5.1 *Let $f : [a, b] \to \mathbf{R}$ be a continuous injective function. Then f is either increasing or decreasing.*

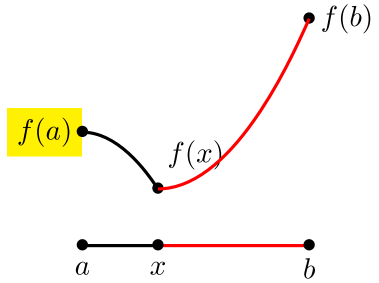
Proof First note that $f(a) \neq f(b)$ by injectivity.

- Step 1. Assume that $f(a) < f(b)$. We first show that if $a < x < b$, then $f(a) < f(x) < f(b)$.

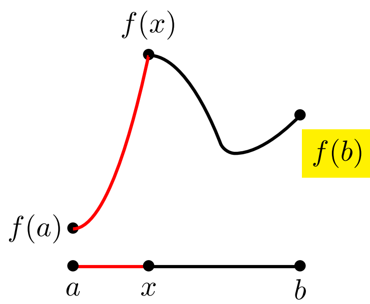
  Note that $f(x) \neq f(a)$ and $f(x) \neq f(b)$ by injectivity. If it is not true that $f(a) < f(x) < f(b)$, then either $f(x) < f(a)$ or $f(b) < f(x)$.



1. In the case where $f(x) < f(a)$, we have $f(x) < f(a) < f(b)$. Take $v = f(a)$. By IVT for f on $[x, b]$, there exists $c \in (x, b)$ with $f(c) = v = f(a)$. Since $c \neq a$, this violates injectivitiy.



2. In the case where $f(b) < f(x)$, we have $f(a) < f(b) < f(x)$. Take $v = f(b)$. By the IVT for f on $[a, x]$, there exists $c \in (a, x)$ with $f(c) = v = f(b)$. This again violates injectivitiy.



- If $f(a) > f(b)$ we show f is decreasing. Let $g = -f$. Then $g(a) < g(b)$. By step 1 g is increasing and so f is decreasing.

□

Theorem 5.2 *If $f : [a, b] \to [f(a), f(b)]$ is increasing and surjective, it is continuous.*

Proof Fix $c \in (a, b)$. Take $\epsilon > 0$. We wish to find the set of x such that $|f(x) - f(c)| < \epsilon$, or

$$f(c) - \epsilon < f(x) < f(c) + \epsilon. \tag{5.1}$$

We may assume that $\epsilon < \min\{f(b) - f(c), f(c) - f(a)\}$. Then

$$f(a) < f(c) - \epsilon < f(c), \qquad f(c) < f(c) + \epsilon < f(b).$$

Since $f : [a, b] \to [f(a), f(b)]$ is surjective we may define

$$a_1 = f^{-1}(f(c) - \epsilon) \quad b_1 = f^{-1}(f(c) + \epsilon).$$

Since f is increasing, (5.1) is equivalent to

$$a_1 < x < b_1.$$

Take $\delta = \min(c - a_1, b_1 - c)$, then $(c-\delta, c+\delta) \subset (a_1, b_1)$. Hence if $|x-c| < \delta$, (5.1) holds. We have proved that f is continuous at $c \in (a,b)$. The continuity of f at a and b can be proved similarly. □

5.4 The Inverse Function Theorem (Continuous Version) – Lecture 10

Suppose that $f : [a,b] \to \mathbf{R}$ is continuous and injective. Then $f : [a,b] \to range(f)$ is a continuous bijection and has inverse $f^{-1} : range(f) \to [a,b]$.

By Theorem 5.1 f is either increasing or decreasing. If it is increasing,

$$f : [a,b] \to [f(a), f(b)]$$

is surjective by the IVT. It has inverse

$$f^{-1} : [f(a), f(b)] \to [a,b]$$

which is also increasing and surjective, and therefore continuous, by 5.2. If f is decreasing,

$$f : [a,b] \to [f(b), f(a)]$$

is surjective, again by the IVT. It has inverse

$$f^{-1} : [f(b), f(a)] \to [a,b]$$

which is also decreasing and surjective, and therefore continuous, again by 5.2.

We have proved

Theorem 5.3 (The Inverse Function Theorem: continuous version) *If $f : [a,b] \to \mathbf{R}$ is continuous and injective, its inverse $f^{-1} : range(f) \to [a,b]$ is continuous.*

The function $f(x) = x^2$ is not injective on all of $\mathbf{R}$. But it is injective on $[0,1]$.

Example 5.1 For $n \in \mathbf{N}$, the function f defined by $f(x) = x^{\frac{1}{n}} : [0,1] \to \mathbf{R}$ is continuous.

Proof Since $x^n : [0,1] \to [0,1]$ is increasing, continuous, its inverse $x^{\frac{1}{n}} : [0,1] \to [0,1]$ is continuous. □

Example 5.2 Show $f(x) = \sqrt{x} : [0,\infty) \to \mathbf{R}$ is continuous.

Proof Let $c \geq 0$. To show f is continuous at $c > 0$. Define

$$g(x) = x^2 : [0, \sqrt{c+1}] \to [0, c+1].$$

Since g is increasing and continuous, its inverse $\sqrt{x} : [0, c+1] \to [0, \sqrt{c+1}]$ is continuous, and is in particular continuous at c. □

Example 5.3 Consider $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$. It is increasing and continuous. Define its inverse $\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then arcsin is continuous.

Remark 5.1 If $f : [a,b] \to [f(a), f(b)]$ is an increasing surjection, its inverse function $f^{-1} : [f(a), f(b)] \to [a,b]$ is continuous.

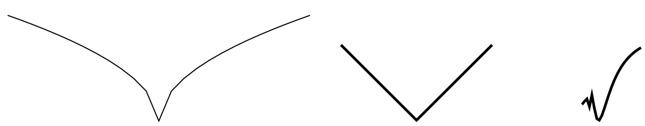
Proof Since $f^{-1} : [f(a), f(b)] \to [a,b]$ is also increasing and surjective. Apply Theorem 5.2 to f^{-1} to see that f^{-1} is continuous. □

Chapter 6

Differentiation of functions

We would like to look at smooth functions. It is reasonable to think that smooth functions correspond to smooth graphs. Think of the graph of $\sin x$ or that of polynomial : they are smooth graphs. A smooth graph should not have cusps and kinks.

These curves are not smooth.



What is a smooth curve exactly? In the first instance imagine that it can be approximated by a straight line at close up. This straight line is the tangent line.

Given a function f, here is the geometric interpretation for $f'(x_0)$: it is the slope of the tangent line at x_0. The tangent line at a point $(x_0, f(x_0))$ is a line that touches the graph of $y = f(x)$ at that point and it approximate the graph in a small neighbourhood of x_0. The value $f(x)$ is close to y, in the linear equation,

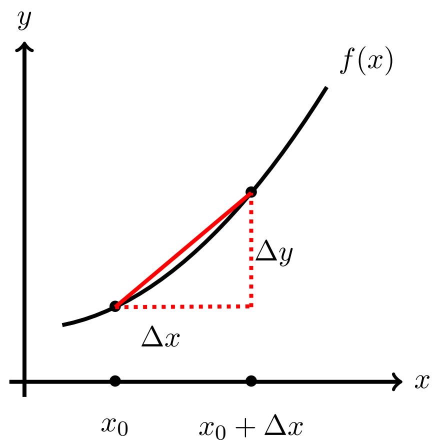
$$y = f(x_0) + f'(x_0)(x - x_0).$$

Let Δx be a number, representing the change in the x-variable. Let Δy denote the corresponding change in y: $\Delta y = f(x_0 + \Delta x) - f(x_0)$.

We define the slope of the tangent to be the limiting slope, of the lines that connecting the two points $(x_0, f(x_0))$ and $(x_0 + \Delta x, f(x_0 + \Delta x))$, as the change Δx approaches 0:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

if the limit exists.



If f is continuous, $\lim_{\Delta x \to 0}[f(x_0 + \Delta x) - f(x_0)] = 0$.
We list some of the motivations for studying derivatives:

- At each point calculate the angle between where two curves intersect. (Descartes)
- Find the local minimums and maximums (Fermat 1638)
- velocity and acceleration of movement (Galilei 1638, Newton 1686).
- verification of some physical laws (Kepler, Newton).
- Determine the shape of the curve given by $y = f(x)$.

6.1 Definition of Differentiability – Lecture 11

Definition 6.1 *Let $x_0 \in (a, b)$ and $f : (a, b) \to \mathbf{R}$. We say that f is differentiable at x_0 if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{6.1}$$

exists (and is a real number, not $\pm\infty$). The limit $f'(x_0)$ is called the derivative of f at x_0 and denoted by $f'(x_0)$.

Note: $\frac{f(x)-f(x_0)}{x-x_0}$ is not defined when $x = x_0$.

Let $h = x - x_0$ and we treat h as a new variable. Then

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}$$

and $h \to 0$ is equivalent to $x \to x_0$. Finally we see that

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}. \tag{6.2}$$

In the following examples we prove differentiability using the $\epsilon - \delta$ definition for differentiability.

Example 6.1 Show that $f(x) = x^2$ is differentiable everywhere.

$$\lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0.$$

Hence $f'(x_0)$ exists and equals $2x_0$.

Example 6.2 Show that $f : \mathbf{R}/\{0\} \to \mathbf{R}$, $f(x) = \frac{1}{x}$ is differentiable at $x_0 \neq 0$.
Proof.

$$\begin{aligned}
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{\frac{1}{x} - \frac{1}{x_0}}{x - x_0} \\
&= \lim_{x \to x_0} \frac{\frac{x_0 - x}{x_0 x}}{x - x_0} \\
&= \lim_{x \to x_0} -\frac{1}{x_0 x} = -\frac{1}{x_0^2},
\end{aligned}$$

in the last step we use that $\frac{1}{x}$ is continuous at $x_0 \neq 0$.

Example 6.3 Show that $\sin x : \mathbf{R} \to \mathbf{R}$ is differentiable everywhere and $(\sin x)' = \cos x$.

Proof Let $x_0 \in \mathbf{R}$.

$$\begin{aligned}
\lim_{h \to 0} \frac{\sin(x_0 + h) - \sin x_0}{h} &= \lim_{h \to 0} \frac{\sin(x_0) \cos h + \cos(x_0) \sin h - \sin x_0}{h} \\
&= \lim_{h \to 0} \frac{\sin(x_0)(\cos h - 1) + \cos(x_0) \sin h}{h} \\
&= \left(\sin(x_0) \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos(x_0) \lim_{h \to 0} \frac{\sin h}{h} \right) \\
&= \sin(x_0) \cdot 0 + \cos(x_0) \cdot 1 = \cos(x_0).
\end{aligned}$$

We used the known limits:

$$\lim_{h\to 0}\frac{(\cos h - 1)}{h} = 0, \qquad \lim_{h\to 0}\frac{\sin h}{h} = 1.$$

□

There are multiple ways to prove the function $|x|$ is not differentiable at $x = 0$. Try it with two sequences converging to 0 while their f values do not converge to the same limit.

Example 6.4 The function $f(x) = |x|$ is not differentiable at $x_0 = 0$.

Proof

$$\begin{aligned}
\lim_{h\to 0+}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0+}\frac{|h|}{h} = \lim_{h\to 0+}\frac{h}{h} = 1,\\
\lim_{h\to 0-}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0-}\frac{|h|}{h} = \lim_{h\to 0-}\frac{-h}{h} = -1.
\end{aligned}$$

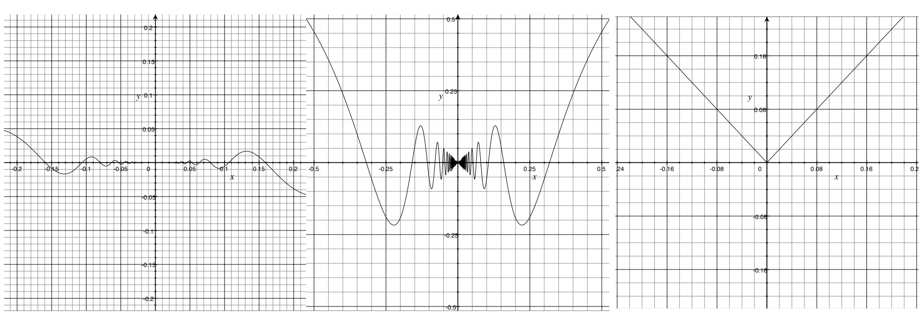
Hence

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

does not exist and f is not differentiable at 0. □

The graph of a differentiable function. The graph of a differentiable function should be smooth: no corners, no cusps.

The graphs below are (from left to right): Graphs of $y = x^2\sin(1/x)$, $y = x\sin(1/x)$ and $y = |x|$. The last two functions are not differentiable at $x = 0$. The first function is differentiable at 0.



Example 6.5 The function

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous everywhere. But it is not differentiable at $x_0 = 0$.
Proof If $x \neq 0$, the function is continuous at x by algebra of continuity. That $f(x)$ is continuous at $x = 0$ is discussed in Example 3.8.

If $x_0 = 0$. Let

$$g(x) = \frac{f(x) - f(0)}{x - 0} = \sin(1/x), \quad x \neq 0.$$

Note that g is defined on $\mathbf{R} - \{0\}$. We have previously proved that $\lim_{x\to 0} g(x)$ does not exist. So f is not differentiable at 0.

[Take $x_n = \frac{1}{2\pi n} \to 0$, $y_n = \frac{1}{2\pi n + \frac{\pi}{2}} \to 0$. Then $g(x_n) = 0$, $g(y_n) = 1$.]

$\square$

Exercise 6.6 Show that

$$f(x) = \begin{cases} x^2\sin(1/x) & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$.

6.2 The Weierstrass-Carathéodory Formulation for Differentiability

Below we give the formulation for the differentiability of f at x_0 given by Carathéodory in 1950.

Theorem 6.1 (**Weierstrass-Carathéodory Formulation**) *Consider $f : (a, b) \to \mathbf{R}$ and $x_0 \in (a, b)$. The following statements are equivalent:*

1. *f is differentiable at x_0*
2. *There is a function ϕ continuous at x_0 such that*

$$f(x) = f(x_0) + \phi(x)(x - x_0). \tag{6.3}$$

Furthermore $f'(x_0) = \phi(x_0)$.

Proof

- 1) $\Rightarrow$ 2) Suppose that f is differentiable at x_0. Set

$$\phi(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0 \end{cases}.$$

  Then
$$f(x) = f(x_0) + \phi(x)(x - x_0).$$
  Since f is differentiable at x_0, ϕ is continuous at x_0.

- 2) $\Rightarrow$ 1). Assume that there is a function ϕ continuous at x_0 such that

$$f(x) = f(x_0) + \phi(x)(x - x_0).$$

  Then
$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} \phi(x) = \phi(x_0).$$
  The last step follows from the continuity of ϕ at x_0. Thus $f'(x_0)$ exists and equal to $\phi(x_0)$.

$\square$

Remark: Compare the above formulation with the geometric interpretation of the derivative. The tangent line at x_0 is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

If f is differentiable at x_0,

$$\begin{aligned} f(x) &= f(x_0) + \phi(x)(x - x_0) \\ &= f(x_0) + \phi(x_0)(x - x_0) + [\phi(x) - \phi(x_0](x - x_0) \\ &= f(x_0) + f'(x_0)(x - x_0) + [\phi(x) - \phi(x_0](x - x_0). \end{aligned}$$

The last step follows from $\phi(x_0) = f'(x_0)$. Observe that

$$\lim_{x\to x_0} [\phi(x) - \phi(x_0)] = 0$$

and so $[\phi(x) - \phi(x_0](x - x_0)$ is insignificant compared to the the first two terms. We may conclude that the tangent line $y = f(x_0) + f'(x_0)(x - x_0)$ is indeed a linear approximation of $f(x)$.

Lecture 12

Corollary 6.2 *If f is differentiable at x_0 then it is continuous at x_0.*

Proof If f is differentiable at x_0, $f(x) = f(x_0) + \phi(x)(x - x_0)$ where ϕ is a function continuous at x_0. By algebra of continuity f is continuous at x_0.

□

Example 6.7 The converse to the Corollary does not hold.

Let $f(x) = |x|$. It is continuous at 0, but fails to be differentiable at 0.

Example 6.8 Consider $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} x^2, & x \in Q \\ 0, & x \notin Q \end{cases}.$$

Claim: f is differentiable only at the point 0.

Proof Take $x_0 \neq 0$. Then f is not continuous at x_0, as we learnt earlier, and so not differentiable at x_0.

Take $x_0 = 0$, let

$$g(x) = \frac{f(x) - f(0)}{x} = \begin{cases} x, & x \in Q \\ 0, & x \notin Q \end{cases}.$$

We learnt earlier that g is continuous at 0 and hence has limit $g(0) = 0$. Thus $f'(0)$ exists and equals 0.

□

Example 6.9 Show that

$$f_1(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$.

Proof Let

$$\phi(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then

$$f_1(x) = \phi(x)x = f_1(0) + \phi(x)(x - 0).$$

Since ϕ is continuous at $x = 0$, f_1 is differentiable at $x = 0$.

[We proved that ϕ is continuous in Example 3.8.] □

Problem 6.3 If a function f is differentiable at x_0 then $f(x) - f(x_0)$ decreases to zero at a rate at least $x - x_0$ when x approaches x_0. If $f'(x_0) = 0$, $f(x) - f(x_0)$ decreases to zero at a rate faster than $x - x_0$. If furthermore we know that f' is differentiable at x_0, $f(x) - f(x_0)$ decreases to zero at a rate at least $(x - x_0)^2$. Denote by $f^{(n)}$ the n-th derivative of f. Set,

$$\phi_2(x) = \begin{cases} \frac{f(x)-f(x_0)}{\frac{1}{2}(x-x_0)^2}, & x \neq x_0 \\ f^{(2)}(x_0), & x = x_0 \end{cases},$$

is ϕ_2 continuous at x_0? Under what assumptions is ϕ_2 a continuous function? You may need L'Hôpital's rule for this. Chanllenge: Make a set of assumptions on f under which $f(x) - f(x_0)$ decays at a rate of $(x - x_0)^n$.

6.3 Properties of Differentiation – Lecture 12

Rules of differentiation can be easily deduced from properties of continuous limits. They can be proved by sequential limit method or even by the basic $\epsilon - \delta$ method if we have the patience.

Theorem 6.4 (Algebra of Differentiability) *Suppose that $x_0 \in (a, b)$ and $f, g : (a, b) \to \mathbf{R}$ are differentiable at x_0.*

1. *Then $f + g$ is differentiable at x_0 and*

$$(f + g)'(x_0) = f'(x_0) + g'(x_0);$$

2. *(product rule) fg is differentiable at x_0 and*

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) = (f'g + fg')(x_0).$$

Proof If f and g are differentiable at x_0 there are two functions ϕ and ψ continuous at x_0 such that

$$f(x) = f(x_0) + \phi(x)(x - x_0) \tag{6.4}$$
$$g(x) = g(x_0) + \psi(x)(x - x_0). \tag{6.5}$$

Furthermore

$$f'(x_0) = \phi(x_0) \qquad g'(x_0) = \psi(x_0).$$

1. Add the two equations (6.4) and (6.5), we obtain

$$\begin{aligned}(f+g)(x) &= f(x_0)+\phi(x)(x-x_0)+g(x_0)+\psi(x)(x-x_0)\\ &= (f+g)(x_0)+[\phi(x)+\psi(x)](x-x_0).\end{aligned}$$

Since $\phi+\psi$ is continuous at x_0, $f+g$ is differentiable at x_0. And

$$(f+g)'(x_0)=(\phi+\psi)(x_0)=f'(x_0)+g'(x_0).$$

2. Multiply the two equations (6.4) and (6.5), we obtain

$$\begin{aligned}(fg)(x) &= \Big(f(x_0)+\phi(x)(x-x_0)\Big)\Big(g(x_0)+\psi(x)(x-x_0)\Big)\\ &= (fg)(x_0)\\ &\quad +\Big(g(x_0)\phi(x)+f(x_0)\psi(x)+\phi(x_0)\psi(x_0)(x-x_0)\Big)(x-x_0).\end{aligned}$$

Let

$$\theta(x)=g(x_0)\phi(x)+f(x_0)\psi(x)+\phi(x_0)\psi(x_0)(x-x_0).$$

Since ϕ,ψ are continuous at x_0, θ is continuous at x_0 by algebra of continuity. It follows that fg is differentiable at x_0. Furthermore

$$(fg)'(x_0)=\theta(x_0)=g(x_0)\phi(x_0)+f(x_0)\psi(x_0)=g(x_0)f'(x_0)+f(x_0)g'(x_0).$$

$\square$

In Lemma 1.4, we showed that if g is continuous at a point x_0 and $g(x_0)\neq 0$, then there is a small neighbourhood of x_0,

$$U^r_{x_0}=(x_0-r,x_0+r)$$

on which $f(x)\neq 0$. Here r is a positive number. This means f/g is well defined on $U^r_{x_0}$ and below in the theorem we only need to consider f restricted to $U^r_{x_0}$.

Theorem 6.5 (Quotient Rule) *Suppose that $x_0\in(a,b)$ and $f,g:(a,b)\to\mathbf{R}$ are differentiable at x_0. Suppose that $g(x_0)\neq 0$, then $\frac{f}{g}$ is differentiable at x_0 and*

$$\left(\frac{f}{g}\right)'(x_0)=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}=\frac{f'g-g'f}{g^2}(x_0).$$

Proof There are two functions ϕ and ψ continuous at x_0 such that

$$\begin{aligned}
f(x) &= f(x_0) + \phi(x)(x - x_0) \\
f'(x_0) &= \phi(x_0) \\
g(x) &= g(x_0) + \psi(x)(x - x_0) \\
g'(x_0) &= \psi(x_0).
\end{aligned}$$

Since g is differentiable at x_0, it is continuous at x_0. By Lemma 1.4,

$$g(x) \neq 0, \qquad x \in (x_0 - r, x_0 + r)$$

where $r > 0$ is a number small so that $(x_0 - r, x_0 + r) \subset (a, b)$. We may divide f by g to see that

$$\begin{aligned}
(\frac{f}{g})(x) &= \frac{f(x_0) + \phi(x)(x - x_0)}{g(x_0) + \psi(x)(x - x_0)} \\
&= \frac{f(x_0)}{g(x_0)} - \frac{f(x_0)[g(x_0) + \psi(x)(x - x_0)]}{g(x_0)[g(x_0) + \psi(x)(x - x_0)]} + \frac{g(x_0)[f(x_0) + \phi(x)(x - x_0)]}{g(x_0)[g(x_0) + \psi(x)(x - x_0)]} \\
&= \frac{f(x_0)}{g(x_0)} + \frac{g(x_0)\phi(x) - f(x_0)\psi(x)}{g(x_0)[g(x_0) + \psi(x)(x - x_0)]}(x - x_0).
\end{aligned}$$

Define

$$\theta(x) = \frac{g(x_0)\phi(x) - f(x_0)\psi(x)}{g(x_0)[g(x_0) + \psi(x)(x - x_0)]}.$$

Since $g(x_0) \neq 0$, θ is continuous at x_0 and (f/g) is differentiable at x_0. And

$$(\frac{f}{g})'(x_0) = \theta(x_0) = \frac{g(x_0)f'(x_0) + f(x_0)g'(x_0)}{[g(x_0)]^2}.$$

$\square$

Lecture 13.

Theorem 6.6 (chain rule) *Let $x_0 \in (a, b)$. Suppose that $f : (a, b) \to (c, d)$ is differentiable at x_0 and $g : (a_1, b_1) \to \mathbf{R}$ is differentiable at $f(x_0)$. Then $g \circ f$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

$$x_0 \xrightarrow{\;f\;} y_0 = f(x_0) \xrightarrow{\;g\;} g(y_0)$$

Proof There is a function ϕ continuous at x_0 such that

$$f(x) = f(x_0) + \phi(x)(x - x_0).$$

Let $y_0 = f(x_0)$. There is a function ψ continuous at y_0 such that

$$g(y) = g(y_0) + \psi(y)(y - y_0).$$

Let $y = f(x)$,

$$\begin{aligned} g\Big(f(x)\Big) &= g(y_0) + \psi\Big(f(x)\Big)\Big(f(x) - f(x_0)\Big) \\ &= g(f(x_0)) + \psi\Big(f(x)\Big)\phi(x)(x - x_0). \end{aligned}$$

Let $\theta(x) = \psi \circ f(x)\phi(x)$. Since f is continuous at x_0 and ψ is continuous at $f(x_0)$, the composition $\psi \circ f$ is continuous at x_0. Then θ, as product of continuous functions, is continuous at x_0. It follows that $f \circ g$ is differentiable at x_0. Since $\phi(x_0) = f'(x_0)$ with $\psi(y_0) = g'(y_0)$ we have

$$(f \circ g)(x_0) = \theta(x_0) = \psi\Big(f(x_0)\Big) \cdot \phi(x_0) = g'\Big(f(x_0)\Big) \cdot f'(x_0).$$

□

Example 6.10 Use the product and chain rules to prove the quotient rule.
Proof Since $g(x_0) \neq 0$, it does not vanish on a neighbourhood of x_0, which denote by

$$U_{x_0}^r := (x_0 - r, x_0 + r).$$

The function f/g is defined everywhere on $U_{x_0}^r$. Let $h(x) = \frac{1}{x}$. Then

$$\frac{f(x)}{g(x)} = f(x)h(g(x)).$$

Since g does not vanish on $U_{x_0}^r$, $h \circ g$ is differentiable at x_0 and $\frac{f}{g} = f \cdot h \circ g$ is differentiable at x_0. For the value of the derivative note that $h'(y) = -\frac{1}{y^2}$,

$$\begin{aligned} (f/g)'(x_0) &= f'(x_0)h(g(x_0)) + f(x_0)h'(g(x_0))g'(x_0) \\ &= \frac{f'(x_0)}{g(x_0)} + f(x_0)\frac{-1}{g^2(x_0)}g'(x_0) \\ &= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}. \end{aligned}$$

The proof is concluded. □

Example 6.11 Use another method to show that if $f, g : (a,b) \to \mathbf{R}$ is differentiable then $f+g$ is differentiabel at x_0.

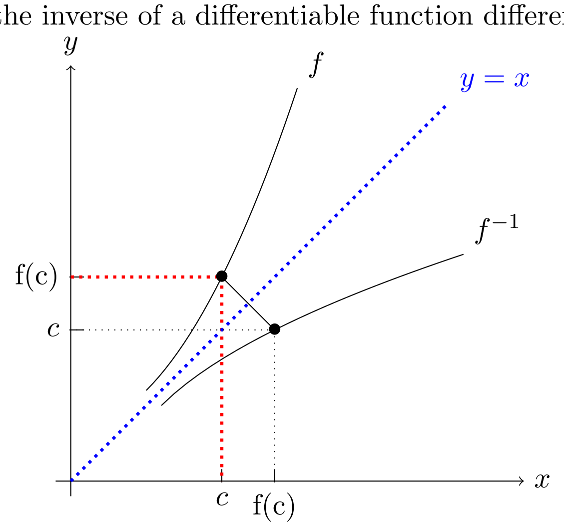
Proof Let $h(x) = f(x) + g(x)$. Then

$$\begin{aligned}
\lim_{x\to x_0} \frac{h(x)-h(x_0)}{x-x_0} &= \lim_{x\to x_0} \frac{f(x)+g(x)-f(x_0)-g(x_0)}{x-x_0} \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} + \lim \frac{g(x)-g(x_0)}{x-x_0} \\
&= f'(x_0) + g'(x_0).
\end{aligned}$$

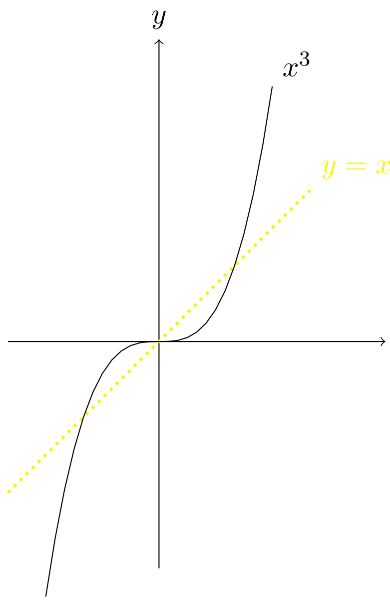
By the definition of differentiability, $f+g$ is differentiable at x_0. □

6.4 The Inverse Function Theorem – Lecture 13

Is the inverse of a differentiable function differentiable?

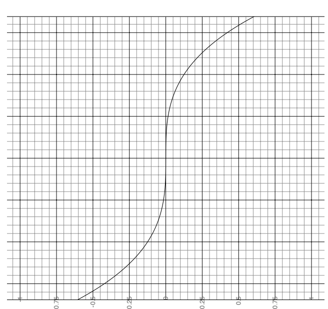


Convince yourself that the graph of f is essentially the graph of f^{-1}, reflected by the line $y = x$. We might believe that f^{-1} is as smooth as f^{-1}, which is essentially correct. However check on the next set of graphs, of which the first if x^3 and the second $x^{1/3}$. Reproduce the graph of the inverse function $x^{\frac{1}{3}}$ from that of x^3 by hand. How does it look like near



zero?

The function $x^{\frac{1}{3}}$ is not differentiable at 0. If one attempts to draw a tangent line, the nearest to it is a vertical line which has infinite slope!



Recall that if f is continuous and injective, the inverse function theorem for continuous functions states that f is either increasing or decreasing. If f is increasing $f : [a, b] \to [f(a), f(b)]$ is a bijection. If f is decreasing $f : [a, b] \to [f(b), f(a)]$ is a bijection. In both cases f^{-1} is continuous.

Theorem 6.7 (The inverse Function Theorem, II) *Let $f : [a, b] \to [c, d]$ be a continuous bijection. Let $x_0 \in (a, b)$ and suppose that f is differentiable at x_0 and $f'(x_0) \neq 0$. Then f^{-1} is differentiable at $y_0 = f(x_0)$. Further-*

more,

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

$$\begin{tikzcd}
\boxed{x_0} \bullet \arrow[r, bend left, "\boxed{f}"] & \bullet\ y_0 \arrow[l, bend left, "f^{-1}"]
\end{tikzcd}$$

Proof Since f is differentiable at x_0,

$$f(x) = f(x_0) + \phi(x)(x - x_0),$$

where ϕ is continuous at x_0. Letting $x = f^{-1}(y)$,

$$f(f^{-1}(y)) = f(f^{-1}(y_0)) + \phi(f^{-1}(y))(f^{-1}(y) - f^{-1}(y_0)).$$

So

$$y - y_0 = \phi(f^{-1}(y))(f^{-1}(y) - f^{-1}(y_0)).$$

Since f is a continuous injective map its inverse f^{-1} in continuous. The composition $\phi \circ f^{-1}$ is continuous at y_0. By the assumption $\phi \circ f^{-1}(y_0) = \phi(x_0) = f'(x_0) \neq 0$, $\phi(x) \neq 0$ for x close to x_0. Define

$$\theta(y) = \frac{1}{\phi(f^{-1}(y))}.$$

It follows that θ is continuous at y_0 and

$$f^{-1}(y) = f^{-1}(y_0) + \theta(y)(y - y_0).$$

Consequently, f^{-1} is differentiable at y_0 and

$$(f^{-1})'(y_0) = \theta(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}.$$

$\square$

Proof [A second proof for the inverse function theorem] A function f is differentiable at x_0 if and only if for all $x_n \in (a,b)\backslash\{x_0\}$ with $\lim_{n\to\infty} x_n = x_0$,

$$f'(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}.$$

By injectivity $f(x_n) \neq f(x_0)$. Since $f'(x_0) \neq 0$,

$$\lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} \to \frac{1}{f'(x_0)}.$$

Take $y_n \in (c,d)$ with $\lim_{n\to\infty} y_n = y_0$. Let $x_n = f^{-1}(y_n)$. Since f^{-1} is continuous,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^{-1}(y_n) = f^{-1}(y_0) = x_0$$

Thus

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{f'(x_0)}.$$

In conclusion, f^{-1} is differentiable at f_0 and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.

$\square$

Comment: The kind of continuous bijective functions we see in everyday life are differentiable everywhere in an interval containing x_0. However there exist continuous bijective functions such that: 1) it is differentiable at x_0 with $f'(x_0) \neq 0$; 2) there exist $x_n \to x_0$ such that f is not differentiable at x_n. Such functions are quite contrived.

Recall the following lemma (Lemma 3.6).

Lemma 6.8 *Let $g : (a,b)/\{c\} \to \mathbf{R}$, and $c \in (a,b)$. Suppose that*

$$\lim_{x\to c} g(x) = l.$$

1. *If $l > 0$, then $g > 0$ close to c.*
2. *If $l < 0$, then $g < 0$ close to c.*

If $f'(x_0) \neq 0$, by the non-vanishing Lemma above, applied to $g(x) = \frac{f(x)-f(x_0)}{x-x_0}$, we may assume that on $(x_0 - r, x_0 + r)$

$$\frac{f(x) - f(x_0)}{x - x_0} > 0.$$

Remark 6.1 1. If $f' > 0$ on an interval $(x_0 - r, x_0 + r)$, by Corollary 7.5 to the Mean Value Theorem in the next chapter, f is an increasing function. Thus

$$f : [x_0 - \frac{r}{2}, x_0 + \frac{r}{2}] \to: [f(x_0 - \frac{r}{2}), f(x_0 + \frac{r}{2})]$$

is a bijection. In summary if $f'(x) > 0$ for all $x \in (a,b)$, then f is invertible.

2. Suppose that f is differentiable on (a, b) and f' is continuous on (a, b). If $x_0 \in (a, b)$ and $f'(x_0) > 0$ then f' is positive on an interval near x_0 and on which the function is invertible.

6.5 One Sided Derivatives

Definition 6.2 *1. A function $f : (a, x_0) \to \mathbf{R}$ is said to have left derivative at x_0 if*

$$\lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The limit shall be denoted by $f'(x_0-)$, called the left derivative of f at x_0.

2. A function $f : (x_0, b) \to \mathbf{R}$ is said to have right derivative at x_0 if

$$\lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The limit shall be denoted by $f'(x_0+)$, called the right derivative of f at x_0.

Theorem 6.9 *A function f has a derivative at x_0 if and only if both $f'(x_0+)$ and $f'(x_0-)$ exist and are equal.*

Example 6.12

$$f(x) = \begin{cases} x \sin(1/x) & x > 0 \\ 0, & x \le 0 \end{cases}$$

Claim: $f'(0-) = 0$ but $f'(0+)$ does not exist.
Proof

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{0 - 0}{x} = 0.$$

But

$$\lim_{x \to 0+} \frac{f(x) - 0}{x - 0} = \lim_{x \to 0+} \sin(1/x)$$

does not exist as can be seen below. Take

$$x_n = \frac{1}{2n\pi} \ge 0, \qquad y_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \ge 0.$$

Both sequences converge to 0 from the right. But

$$\sin(1/x_n) = 0, \qquad \sin(y_n) = 1.$$

They have different limits so $\lim_{x\to 0+} \sin(1/x)$ does not exist. Thus $f'(0+)$ does not exist.

$\square$

Example 6.13

$$g(x) = \begin{cases} \sqrt{x}\sin(1/x) & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Claim: $g'(0+)$ does not exist.
Proof Note that

$$\lim_{x\to 0+} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0+} \frac{1}{\sqrt{x}} \sin(1/x).$$

Take

$$y_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \to 0.$$

But

$$\frac{1}{\sqrt{y_n}} \sin(1/y_n) = \sqrt{\frac{\pi}{2} + 2n\pi} \to +\infty.$$

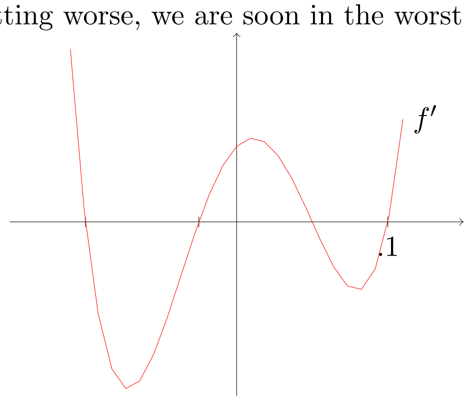
Hence $\frac{1}{\sqrt{x}}\sin(1/x)$ cannot converge to a finite number as x approaches 0 from the right. We conclude that $g'(0+)$ does not exist. $\square$

Chapter 7

The mean value theorem

If we have some information on the derivative of a function what can we say about the function itself?

What do we think when our minister tells us that "The rate of inflation is slowing down"? This means that if f is the price of a commodity as a function of t, its derivative is decreasing, not necessarily the price itself (in fact most surely not, otherwise we would be told so directly)! I heard the following on the radio: "the current trend of increasing unemployment is slowing down". What does it really mean? How about 'the bad time is getting worse, we are soon in the worst spot and out of it'?



Could you sketch a plausible graph of f from the graph of f?

Find critical points (where $f' = 0$). Identify the critical points to see whether it is a local maximum or a local minimum. If f' is increasing it is a local minimum. Identify inflection points (where f' reaches local minimum or local maximum). At these points the graph change convexity. Identify whether the graphs between the critical points are increasing or decreasing, convex or concave. Now sketch the graph of f.

7.1 Local Extrema – Lecture 14

Definition 7.1 *Consider* $f : [a, b] \to \mathbf{R}$ *and* $x_0 \in [a, b]$.

1. *We also say that f has a local maximum at* x_0, *if there is a deleted neighbourhood* $(x_0 - r, x_0 + r)\backslash\{x_0\}$ *of* x_0 *on which* $f(x_0) \ge f(x)$. *Here* $r > 0$. *We also say that* x_0 *is a local maximal point.*

   If $f(x_0) > f(x)$, *we may say that* f *has a strict local maximum at* x_0.

2. *We also say that f has a local minimum at* x_0, *if for some* $r > 0$,

$$f(x_0) \le f(x), \qquad x \in (x_0 - \delta, x_0 + \delta)\backslash\{x_0\}.$$

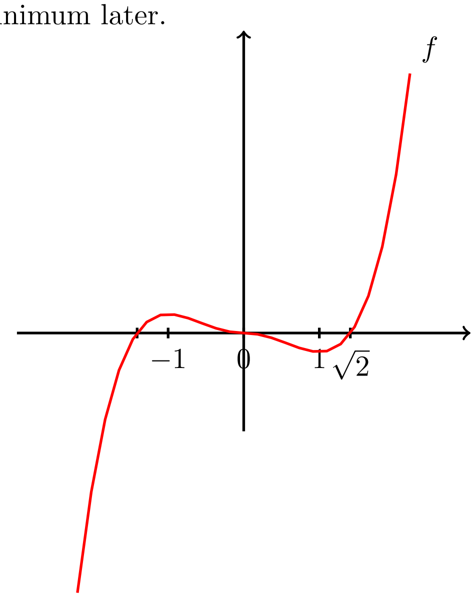
3. *We also say that f have a local extremum at* x_0 *if it either has a local minimum or a local maximum at* x_0.

Example 7.1 Let $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$. Then

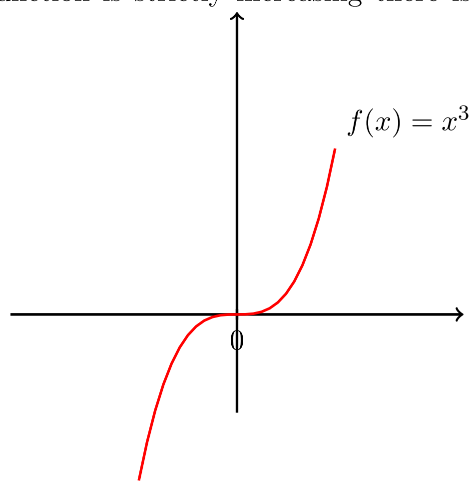
$$f(x) = \frac{1}{4}x^2(x^2 - 2).$$

It has a local maximum at $x = 0$, since $f(0) = 0$ and near $x = 0$, $f(x) < 0$.

It has local minimum at $x = \pm 1$. We could show that they are local minimum later.



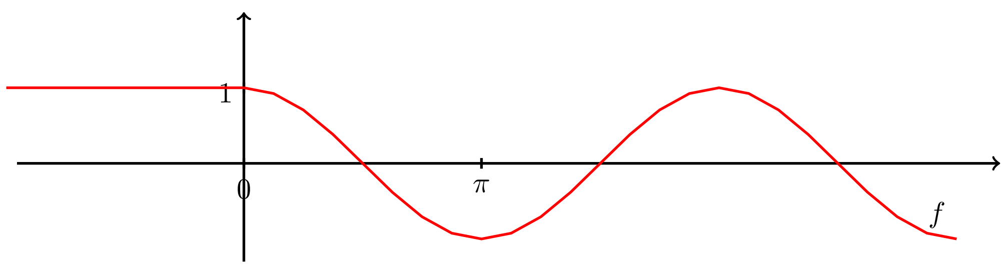
Example 7.2 $f(x) = x^3$. This function is strictly increasing there is no



local minimum or local maximum.

Example 7.3

$$f(x) = \begin{cases} 1, & x \in [-\pi, 0], \\ \cos(x), & x \in [0, 3\pi]. \end{cases}.$$



Then $x = 0$ is a local maximum, $x = 2\pi$ is a strict local maximum, and $x = -\pi$ is a strict local minimum.

Lemma 7.1 *Consider $f : (a, b) \to \mathbf{R}$. Suppose that $x_0 \in (a, b)$ is either a local minimum or a local maximum of f. If f is differentiable at x_0 then $f'(x_0) = 0$.*

Proof Case 1. We first assume that f has a local maximum at x_0:

$$f(x_0) \ge f(x), \qquad x \in (x_0 - \delta_1, x_0 + \delta_1),$$

for some $\delta_1 > 0$. Then

$$f'_+(x_0) = \lim_{h \to 0+} \frac{f(x_0 + h) - f(x_0)}{h} \le 0$$

since $h \in (0, \delta_1)$ eventually. Similarly

$$f'_-(x_0) = \lim_{h \to 0-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

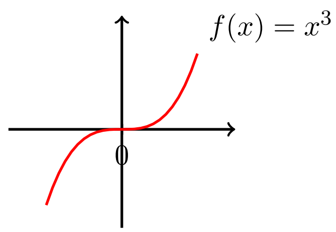
Since the derivative exists we must have

$$f'(x_0) = f'_+(x_0) = f'_-(x_0).$$

Finally we deduce that $f'(x_0) = 0$.

Case 2. If x_0 is a local minimum of f. Take $g = -f$. Then x_0 is a local maximum of g and $g'(x_0) = -f'(x_0)$ exists. It follows that $g'(x_0) = 0$ and consequently $f'(x_0) = 0$. □

Example 7.4 However $f'(x_0) = 0$ does not necessarily imply that x_0 is a local extrema. Consider $f : -[-1, 1]$ and $f(x) = x^3$. Then $f'(0) = 0$, but f does not have a local minimum nor a local maximum at 0.



7.2 Global Maximum and Minimum

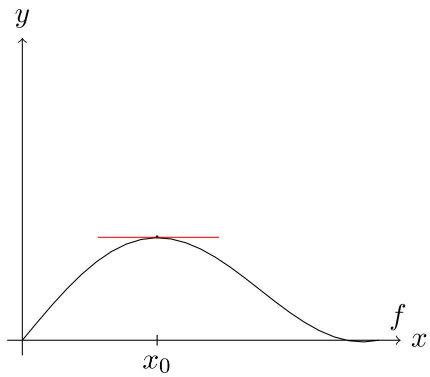
If $f : [a, b] \to \mathbf{R}$ is continuous by the extreme value theorem, it has a minimal and a maximal point. How do we find the maximum and minimum?

What we learnt in the last section suggested that we find all critical points.

Definition 7.2 *A point c is a critical point of f if either $f'(c) = 0$ or $f'(c)$ does not exist.*

To find the (global) maximum and minimum of f, evaluate the values of f at a, b and all the critical values. Select the largest value.

7.3 Rolle's Theorem



Theorem 7.2 (Rolle's Theorem) *Suppose that*

1. *f is continuous on $[a, b]$*
2. *f is differentiable on (a, b)*
3. *$f(a) = f(b)$.*

Then there is a point $x_0 \in (a, b)$ such that

$$f'(x_0) = 0.$$

Proof If f is a constant, Rolle's Theorem holds.

Otherwise by the Extreme Value Theorem, there are points $\underline{x}, \bar{x} \in [a, b]$ with

$$f(\bar{x}) \le f(x) \le f(\bar{x}), \quad \forall x \in [a, b].$$

Since f is not a constant $f(\underline{x}) \ne f(\bar{x})$.

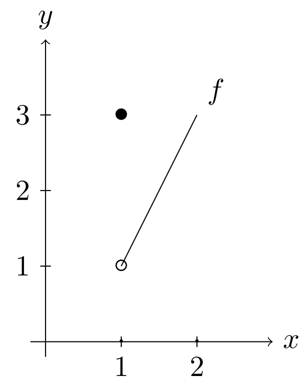
Since $f(a) = f(b)$, one of the point $\bar{x}$ or $\underline{x}$ is in the open interval (a, b). Denote this point by x_0. By the previous lemma $f'(x_0) = 0$. $\square$

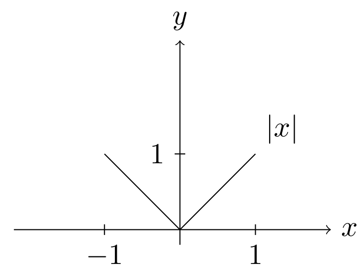
Example 7.5 *Discussions on Assumptions:*

1. Continuity on the closed interval $[a, b]$ is necessary. e.g. consider $f : [1, 2] \to \mathbf{R}$.

$$f(x) = \begin{cases} f(x) = 2x - 1, & x \in (1, 2] \\ f(1) = 3, & x = 1. \end{cases}$$

The function f is continuous on $(1, 2]$ and is differentiable on (a, b). Also $f(2) = 3 = f(1)$. But there is no point in $x_0 \in (1, 2)$ such that $f'(x_0) = 0$.



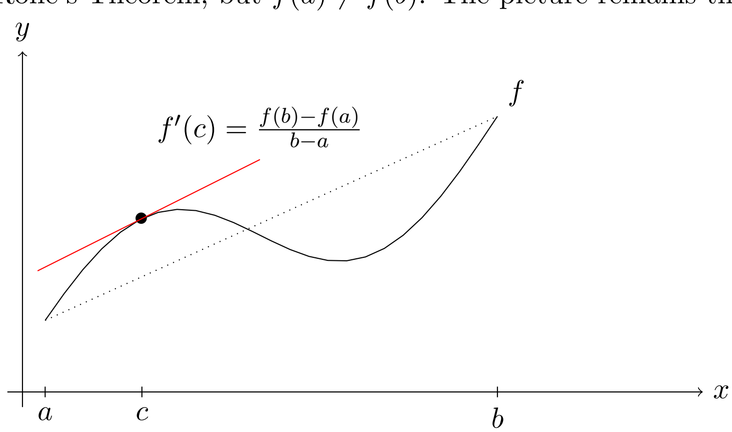


2. Differentiability is necessary. e.g. take $f : [-1, 1] \to \mathbf{R}$, $f(x) = |x|$. Then f is continuous on $[-1, 1]$, $f(-1) = f(1)$. But there is no point on $x_0 \in (-1, 1)$ with $f'(x_0) = 0$. This point ought to be $x = 0$ but the function fails to be differentiable at x_0.

Exercise 7.6 Let $f(x) = x \sin x - \frac{1}{2}x - \sin x + \frac{1}{2}$. Does f have a critical point on $(\frac{\pi}{6}, 1)$?

7.4 The Mean Value Theorem – Lecture 15

Let us now tilde the graph of a function f satisfying the regularity conditions of Rolle's Theorem, but $f(a) \neq f(b)$. The picture remains the 'same'.



Theorem 7.3 (The Mean Value Theorem) *Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b), then there is a point $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof Set

$$k = \frac{f(b) - f(a)}{b - a}$$

and consider the line equation connecting $(a, f(a))$ and $(b, f(b))$:

$$h(x) = f(a) + k(x - a).$$

Set

$$g(x) = f(x) - h(x)$$

Then $g(b) = g(a) = 0$. Now g is continuous on $[a,b]$ and differentiable on (a,b). Apply Rolle's Theorem to g on $[a,b]$: there is $c \in (a,b)$ with $g'(c) = 0$. THe required result follows from

$$g'(c) = f'(c) - k = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

□

The x-co-ordinates of the points where g intersects the x-axis are the x-co-ordinates of the points f intersects the line L. The x-coordinates where the tangent line to f have slope k are the x-co-ordinates of the local extrama of g and the tangent slope to g are zero at these points.

Corollary 7.4 *Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b). Suppose that $f'(x) = 0$, for $x \in (a,b)$. Then $f(x)$ is constant on $[a,b]$.*

Proof Let for $x \in (a,b]$. By the mean value theorem on $[a,x]$, there is $c \in (a,x)$ with

$$\frac{f(x) - f(a)}{x - a} = f'(c) = 0.$$

Hence $f(x) = f(a)$. □

Example 7.7 Claim: if $f'(x) = 1$ then $f(x) = x + c$.

Let $g(x) = f(x) - x$. Then $g'(x) = 0$ Hence $g(x)$ is a constant: $g(x) = g(a) = f(a) - a$. Consequently $f(x) = x + g(x) = f(a) + x - a$.

Exercise 7.8 If $f'(x) = x$. What does f look like?

Exercise 7.9 Show that if f, g are continuous on $[a, b]$ and differentiable on (a, b), and $f' = g'$ on (a, b), then $f = g + C$ where $C = f(a) - g(a)$.

Remark 7.1
- If f' is continuous on $[a, b]$, it is integrable. Since f' is continuous, by the Extreme Value Theorem, there are $x_1, x_2 \in [a, b]$ such that $m = f'(x_1)$ and $M = f'(x_2)$ are respectively the minimum and the maximum.

  However, by the following inequality borrowed from Analysis III,

$$m(b-a) \le \int_a^b f'(x)dx \le M(b-a).$$

  Divide by $b - a$ to see that

$$m \le \frac{1}{b-a} \int_a^b f'(x)dx \le M.$$

  Apply the Intermediate Value Theorem to f' on $[x_1, x_2]$. There is $c \in [x_1, x_2]$ such that

$$f'(c) = \frac{1}{b-a} \int_a^b f'(x)dx.$$

  This says that the mean gradient is attained at c.

- By the fundamental theorem of calculus, again borrowed from Analysis III,

$$\int_a^b f'(x) = f(b) - f(a).$$

  Hence when f' is continuous, we deduced the Mean Value Theorem from the Intermediate Value Theorem.

7.5 Monotonicity and Derivatives

N.B. If $f : [a, b] \to \mathbf{R}$ is increasing and differentiable at x_0, then $f'(x_0) \ge 0$. This is because

$$f'(x_0) = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \ge 0.$$

Example 7.10 Let $f(x) = x^3$. Its derivative $f'(x) = 2x^2 \ge 0$.

Corollary 7.5 *Suppose that $f : R \to R$ is continuous on $[a, b]$ and differentiable on (a, b).*

1. *If $f'(x) \ge 0$ for all $x \in (a, b)$, then f is non-decreasing on $[a, b]$.*
2. *If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.*
3. *If $f'(x) \le 0$ for all $x \in (a, b)$, then f is non-increasing on $[a, b]$.*
4. *If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.*

Proof Part 1. Suppose $f'(x) \ge 0$ on (a, b). Take $x, y \in [a, b]$ with $x < y$. By the mean value applied on $[x, y]$, there exists $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

and

$$f(y) - f(x) = f'(c)(y - x) \ge 0.$$

Thus $f(y) \ge f(x)$.

Part 2. If $f'(x) > 0$ everywhere then as in part 1, for $x < y$,

$$f(y) - f(x) = f'(c)(y - x) > 0.$$

□

Remark 7.2 (Discussion on the assumptions.) Let f be defined on an interval containing (a, b).

1. That f is increasing on (a, b) follows if f has the same sign on the whole interval (a, b). If the derivative of f is positive at one point x_0, it does not mean the function is increasing on any interval that containing x_0. This could happen if f' is not continuous. See the next example.

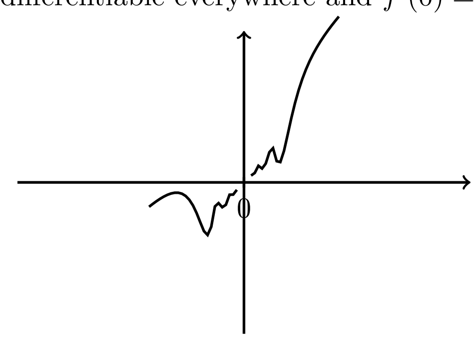
   However we have the following: Suppose that $f'(x_0) > 0$. Then for x close to x_0, i.e. $x \in (x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$,

   $$f(x) < f(x_0) \text{ if } x < x_0 \text{ and } \quad f(x) > f(x_0) \text{ if } x > x_0.$$

   Explain why this is not the same as the statement ' f is increasing on $(x_0 - \delta, x_0 + \delta)$'?

2. Assume that $f'(x_0) > 0$. Although we cannot compare $f(x), f(y)$ for two arbitrary x, y in the neighbourhood of x_0, we can always compare $f(x_0)$ with $f(x)$ for x near x_0. In fact, since
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) > 0,$$
by Lemma 3.6,
$$\frac{f(x) - f(x_0)}{x - x_0} > 0$$
on a small neighbourhood U of x_0. Hence on this neighbourhood,
$$\begin{array}{llcl} f(x) > & f(x_0), & & \text{if } x > x_0 \\ f(x) < & f(x_0), & & \text{if } x > x_0. \end{array}$$

Example 7.11 The function
$$f(x) = \begin{cases} x + x^2 \sin(\frac{1}{x^2}), & x \neq 0, x \in [-1, 1] \\ 0, & x = 0 \end{cases}.$$
is differentiable everywhere and $f'(0) = 1$.



Proof Since the sum, product, composition and quotient of differentiable functions are differentiable, *provided the denominator is not zero*, f is differentiable at $x \neq 0$, and $f'(x) = 1 + 2x \sin(\frac{1}{x^2}) - \frac{2}{x} \cos(\frac{1}{x^2})$. At $x = 0$,
$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} \\ &= \lim_{x \to 0} \frac{x + x^2 \sin(\frac{1}{x^2})}{x} \\ &= \lim_{x \to 0} [1 + x \sin(\frac{1}{x^2})] = 1. \end{aligned}$$

$\square$ Notice, however, although f is everywhere differentiable, the function $f'(x)$ is not continuous at $x = 0$.

Example 7.12 Take $f : [-1,1] \to \mathbf{R}$.

$$f(x) = \begin{cases} x + x^2 \sin(\frac{1}{x^2}), & x \neq 0, x \in [-1,1] \\ 0, & x = 0 \end{cases}.$$

Claim: Even though $f'(0) = 1 > 0$, there is no interval $[-\delta, \delta]$ on which f is increasing.

Proof

$$f'(x) = \begin{cases} 1 + 2x \sin(\frac{1}{x^2}) - \frac{2}{x} \cos(\frac{1}{x^2}), & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Consider the intervals:

$$I_n = \left[\frac{1}{\sqrt{2\pi n + \frac{\pi}{4}}}, \frac{1}{\sqrt{2\pi n}} \right].$$

If $x \in I_n$, $\sqrt{2\pi n} \le \frac{1}{x} \le \sqrt{2\pi n + \frac{\pi}{4}}$ and

$$\cos \frac{1}{x^2} \ge \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Thus if $x \in I_n$

$$f'(x) \le 1 + 2\frac{1}{\sqrt{2\pi n}} - 2\sqrt{2\pi n}\frac{1}{\sqrt{2}} = \frac{\sqrt{\pi n} + \sqrt{2} - 2\pi n}{\sqrt{\pi n}} < 0.$$

Consequently f is decreasing on I_n. As every open neighbourhood of 0 contains an interval I_n, the claim is proved.

□

Example 7.13 The function f, defined above, has infinitely many local minima and local maxima near zero.

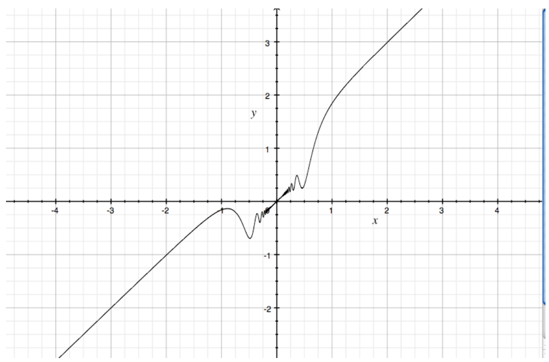
Proof Let

$$J_n = \left[\frac{1}{\sqrt{2\pi n + \pi}}, \frac{1}{\sqrt{\frac{2\pi n + \pi}{2}}} \right].$$

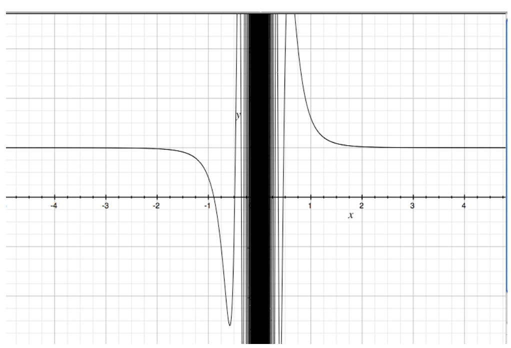
An argument like that of the preceding proof shows that f is strictly increasing on J_n. Since f is strictly increasing on J_n and strictly decreasing on I_n, it follows that in

$$\left(\frac{1}{\sqrt{\frac{2\pi n + \pi}{2}}}, \frac{1}{\sqrt{2\pi n + \frac{\pi}{4}}} \right)$$

, i.e. between the right hand end of I_n and the left hand end of J_n f must have a local maximum, and between the right hand end of I_n and the left hand end of J_{n-1} f must have a local minimum. $\square$



$y = x + x^2 \sin\left(\frac{1}{x^2}\right)$



$y = 1 + 2x\sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right)$

7.6 Inequalities From the MVT – Lecture 16

We wish to use the information on f' to deduce behaviour of f. Suppose we have a bound for f' on the interval $[a, x]$. Applying the mean value theorem to the interval $[a, x]$, we have

$$\frac{f(x) - f(a)}{x - a} = f'(c)$$

for some $c \in (a, x)$. From the bound on f' we can deduce something about $f(x)$.

Example 7.14 For any $x > 1$,

$$1 - \frac{1}{x} < \ln(x) < x - 1.$$

Proof Consider the function ln on $[1, x]$. We will prove later that $\ln : (0, \infty) \to \mathbf{R}$ is differentiable everywhere and $\ln'(x) = \frac{1}{x}$. We already know for any $b > 0$, $\ln : [1, b] \to \mathbf{R}$ is continuous, as it is the inverse of the continuous function $e^x : [0, \ln b] \to [1, b]$.

Fix $x > 0$. By the mean value theorem applied to the function $f(y) = \ln y$ on the interval $[0, x]$, there exists $c \in (1, x)$ such that

$$\frac{\ln(x) - \ln 1}{x - 1} = f'(c) = \frac{1}{c}.$$

Since $\frac{1}{x} < \frac{1}{c} < 1$,

$$\frac{1}{x} < \frac{\ln x}{x - 1} < 1.$$

Multiplying through by $x - 1 > 0$, we see

$$\frac{x - 1}{x} < \ln x < x - 1.$$

$\square$

Example 7.15 Let $f(x) = 1 + 2x + \frac{2}{x}$. Show that $f(x) \le 23.1 + 2x$ on $[0.1, \infty)$.

Proof First $f'(x) = 2 - \frac{2}{x^2}$. Since $f'(x) < 0$ on $[0.1, 1)$, by Corollary 7.5 of the MVT, f decreases on $[0.1, 1]$. So the maximum value of f on $[0.1, 1]$ is $f(0.1) = 23.1$. On $[1, \infty)$, $f(x) \le 1 + 2x + 2 = 3 + 2x$. $\square$

7.7 Asymptotics of f at Infinity – Lecture 16

Example 7.16 Let $f : [1, \infty) \to \mathbf{R}$ be continuous. Suppose that it is differentiable on $(1, \infty)$. If $\lim_{x \to +\infty} f'(x) = 1$ there is a constant K such that

$$f(x) \le 2x + K.$$

Proof

1. Since $\lim_{x \to +\infty} f'(x) = 1$, taking $\epsilon = 1$, there is a real number $A > 1$ such that if $x > A$, $|f'(x) - 1| < 1$. Hence

   $$f'(x) < 2 \qquad \text{if } x > A.$$

   We now split the interval $[1, \infty)$ into $[1, A] \cup (A, \infty)$ and consider f on each interval separately.

2. Case 1: $x \in [1, A]$. By the Extreme Value Theorem, f has an upper bound K on $[1, A]$. If $x \in [1, A]$, $f(x) \le K$. Since $x > 0$,

   $$f(x) \le K + 2x, \qquad x \in [1, A].$$

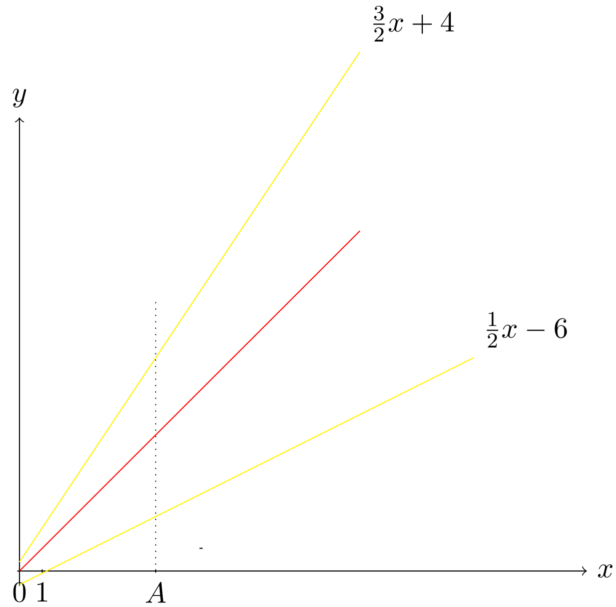
3. Case 2. $x > A$. By the Mean Value Theorem, applied to f on $[A, x]$, there exists $c \in (A, x)$ such that

   $$\frac{f(x) - f(A)}{x - A} = f'(c) < 2,$$

   by part 1). So if $x > A$,

   $$f(x) \le f(A) + 2(x - A) \le f(A) + 2x \le K + 2x.$$

The required conclusion follows. $\square$



The graph of f should fall into the wedge with the yellow boundaries.

Exercise 7.17 Let $f(x) = x + \sin(\sqrt{x})$. Find $\lim_{x\to+\infty} f'(x)$.

Exercise 7.18 In Example 7.16, can you find a lower bound for f?

Example 7.19 Let $f : [1, \infty) \to \mathbf{R}$ be continuous. Suppose that it is differentiable on $(1, \infty)$. Show that if $\lim_{x\to+\infty} f'(x) = 1$ then there are real numbers m and M such that for all $x \in [1, \infty)$,

$$m + \frac{1}{2}x \le f(x) \le M + \frac{3}{2}x.$$

Proof

1. By assumption $\lim_{x\to+\infty} f'(x) = 1$. For $\epsilon = \frac{1}{2}$, there exists $A > 1$ such that if $x > A$,
$$\frac{1}{2} = 1 - \epsilon < f'(x) < 1 + \epsilon = \frac{3}{2}.$$

2. Let $x > A$. By the Mean Value Theorem, applied to f on $[A, x]$, there exists $c \in (A, x)$ such that
$$f'(c) = \frac{f(x) - f(A)}{x - A}.$$

Hence
$$\frac{1}{2} < \frac{f(x) - f(A)}{x - A} < \frac{3}{2}.$$
So if $x > A$,
$$\begin{aligned} f(A) + \frac{1}{2}(x - A) &\le f(x) \le & f(A) + \frac{3}{2}(x - A) \\ f(A) - \frac{1}{2}A + \frac{1}{2}x &\le f(x) \le & f(A) + \frac{3}{2}x. \end{aligned}$$

3. On the finite interval $[1, A]$, we may apply the Extreme Value Theorem to $f(x) - \frac{1}{2}x$ and to $f(x)$ so there are m, M such that
$$m \le f(x) - \frac{1}{2}x, \qquad f(x) \le M.$$
Then if $x \in [1, A]$,
$$\begin{aligned} f(x) &\le M \le M + \frac{3}{2}x \\ f(x) &= f(x) - \frac{1}{2}x + \frac{1}{2}x \ge m + \frac{1}{2}x. \end{aligned}$$

4. Since $m \le f(A) - \frac{1}{2}A$, the required identity also holds if $x > A$.

$\square$

Exercise 7.20 In Example 7.19 can you show that
$$m + 0.09x \le f(x) \le 1.01x + M?$$

Exercise 7.21 Let $f : [1, \infty) \to \mathbf{R}$ be continuous. Suppose that it is differentiable on $(1, \infty)$. Suppose that $\lim_{x\to+\infty} f'(x) = k$. Show that for any $\epsilon > 0$ there are numbers m, M such that
$$m + (k - \epsilon)x \le f(x) \le M + (k + \epsilon)x.$$

Comment: If $\lim_{x\to+\infty} f'(x) = k$, the graph of f lies in the wedge formed by the lines $y = M + (k + \epsilon)x$ and $y = m + (k - \epsilon)x$. This wedge contains and closely follows the line with slope k. But we have to allow for some oscillation and hence the ϵ, m and M. In the same way if $|f'(x)| \le Cx^\alpha$ when x is sufficiently large then $f(x)$ is controlled by $x^{1+\alpha}$ with allowance for the oscillation.

Exercise 7.22 What happens if $\lim_{x\to\infty} f(x) = 0$? Could we say that f cannot grow more than sub-linearly?

Let us first consider the monotone functions $f(x) = x^\alpha$.

Example 7.23 Let $f : (0,\infty) \to \mathbf{R}$, $f(x) = x^\alpha$, $x > 0$. Then $f'(x) = \alpha x^{\alpha-1}$ and

$$\lim_{x\to+\infty} f(x) = \begin{cases} +\infty, & \text{if } \alpha > 0 \\ 1, & \text{if } \alpha = 0 \\ 0, & \text{if } \alpha < 0. \end{cases}$$

$$\lim_{x\to+\infty} f'(x) = \begin{cases} +\infty, & \text{if } \alpha > 1 \\ 1, & \text{if } \alpha = 1 \\ 0, & \text{if } \alpha < 1. \end{cases}$$

If $\alpha < 1$ the function x^α grows sub-linearly at x.
Let us now add some oscillation. How do we do this?

Example 7.24 Consider

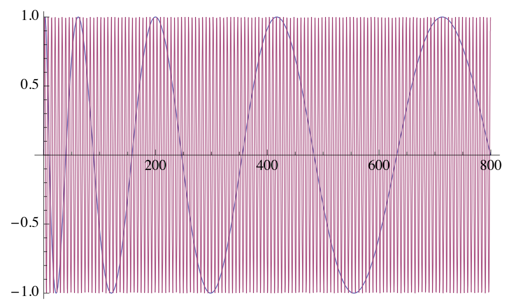
$$f(x) = x^\alpha + \sin x.$$

The function f is dominated by x^α when $\alpha > 0$ and x goes to $+\infty$. For $\alpha < 0$, as x goes to $+\infty$ it behaves like $\sin x$ and oscillates between -1 and 1.

The graph of sin (sqrt (x))) and sin (x) on [0, 81 Pi^2] (zoomed out). The pink graph is that of sin x. Note that how it oscillates compare to that of sin (sqrt (x)).

In[47]:=
```
Plot[{Sin[x^(1/2)], Sin[x]}, {x, 0, (9 Pi)^2}]
```

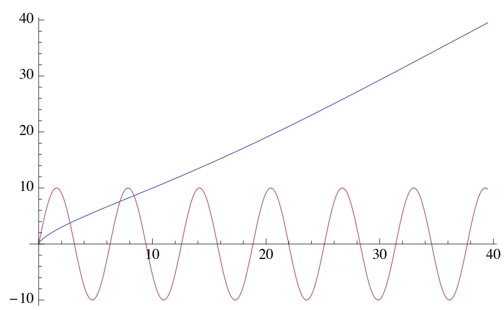
Out[47]=



The graph looking like a straight line is that of x + sin (sqrt (x))

In[44]:=
```
Plot[{x + Sin[x^(1/2)], 10 * Sin[x]}, {x, 0, (2 Pi)^2}]
```

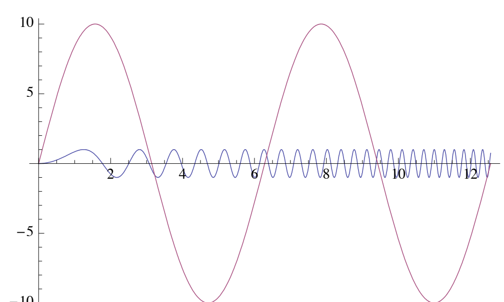
Out[44]=



The fast oscillation graph is that of sin (x^2).

In[43]:=
```
Plot[{Sin[x^(2)], 10 * Sin[x]}, {x, 0, 4 Pi}]
```

Out[43]=



Example 7.25 Let $\beta > 0$ and $f : (0, \infty) \to \mathbf{R}$,

$$f(x) = \sin(\frac{1}{x^\beta}).$$

When x is large, we do not see oscillation from the sine function. Indeed if $x > (\frac{2}{\pi})^{\frac{1}{\beta}}$, then $0 < \frac{1}{x^\beta} < \frac{\pi}{2}$, and $\sin \frac{1}{x^\beta}$ is increasing.

Example 7.26 Let $\beta > 0$ and $f : (0, \infty) \to \mathbf{R}$,

$$f(x) = x^\alpha \sin(\frac{1}{x^\beta}).$$

We know that $\lim_{x\to\infty} \sin(\frac{1}{x}) = 0$; how about $\lim_{x\to\infty} x \sin \frac{1}{x}$? Indeed for any $\beta > 0$, let $y = \frac{1}{x^\beta}$. We see

$$\lim_{x\to+\infty} x^\beta \sin(\frac{1}{x^\beta}) = \lim_{y\to 0+} \frac{\sin(y)}{y} = 1.$$

Hence

$$\lim_{x\to+\infty} x^\alpha \sin(\frac{1}{x^\beta}) = \lim_{x\to+\infty} x^{\alpha-\beta} \left[x^\beta \sin(\frac{1}{x^\beta}) \right].$$

As x goes to ∞, the function f behaves like $x^{\alpha-\beta}$.

Example 7.27 Let $f(x) = \sqrt{x} \sin(\frac{1}{x^{\frac{1}{4}}})$. Then $f(x)$ does not have a limit as x goes to ∞, and is not even bounded. Nevertheless

$$\lim_{x\to+\infty} f'(x) = \lim_{x\to+\infty} \left(\frac{1}{2\sqrt{x}} \sin(\frac{1}{x^{\frac{1}{4}}}) - \frac{1}{4} \frac{1}{x^{\frac{3}{4}}} \cos(\frac{1}{x^{\frac{1}{4}}}) \right) = 0.$$

Let us now look at a more interesting oscillating function.

Example 7.28 Let $\beta > 0$. Let $f : (0, \infty) \to \mathbf{R}$,

$$f(x) = x^\alpha \sin(x^\beta).$$

1. The bigger is β, the more quickly x^β grows (once $x > 1$). The more quickly x^β grows, the more quickly $\sin(x^\beta)$ oscillates. So large β means faster oscillation.

2. If $\alpha < 0$, $\lim_{x\to+\infty} f(x) = 0$; if $\alpha \ge 0$, $\lim_{x\to+\infty} f(x)$ does not exist. To see the conclusion for $\alpha \ge 0$ take $x_n = (2\pi n)^{\frac{1}{\beta}} \to \infty$, then

$$\lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} x_n^\alpha \cdot 0 = 0.$$

Take $y_n = (2\pi n + \frac{\pi}{2})^{\frac{1}{\beta}} \to \infty$, then

$$\lim_{n\to+\infty} f(y_n) = \lim_{n\to\infty} y_n^{\alpha} \cdot 1 = (2\pi n + \frac{\pi}{2})^{\frac{\alpha}{\beta}} = \begin{cases} \infty, & \text{if } \alpha > 0; \\ 1, & \text{if } \alpha = 0. \end{cases}$$

So $\lim_{n\to\infty} f(y_n) \neq \lim_{n\to+\infty} f(x_n)$.

3.
$$f'(x) = \alpha x^{\alpha-1}\sin(x^{\beta}) + x^{\alpha+\beta-1}\cos(x^{\beta}).$$

Note that $\beta > 0$ and so $\alpha - 1 < \alpha + \beta - 1$ and $x^{\alpha+\beta-1}$ dominates $x^{\alpha-1}$

(a) Suppose that $\alpha + \beta \geq 1$, then $\lim_{x\to\infty} f'(x)$ does not exist.

(b) Suppose that $\alpha + \beta < 1$. Since $\alpha < 1$,

$$\lim_{x\to+\infty} f'(x) = 0.$$

Example 7.29 In each of the following three cases, evaluate $lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} f'(x)$.

- Take $\alpha = 0, \beta = \frac{3}{4}$, $f(x) = \sin(x^{\frac{3}{4}})$.
- Take $\alpha = \frac{1}{2}, \beta = \frac{1}{4}$, $f(x) = \sqrt{x}\sin(x^{\frac{1}{4}})$.
- Take $\alpha = 1, \beta = \frac{1}{4}$, $f(x) = x\sin(x^{\frac{1}{4}})$.

Example 7.30 Consider $g(x) = 1 + \frac{1}{x}\sin x$.

$$\begin{aligned} \lim_{x\to+\infty} g(x) &= 1 \\ \lim_{x\to+\infty} g'(x) &= \lim_{x\to+\infty} \left(-\frac{1}{x^2}\sin(x) + \frac{1}{x}\cos(x)\right) = 0. \end{aligned}$$

7.8 An Example: Gaussian Envelopes

I do not cover this section in class. However it is a fun example. Do read on! Let $f_\sigma(x), \sigma \in I$ be a family of continuous functions with index σ in a set I. We say that f is parametrized by σ. We seek a function $g(x)$ such that for each x, $g(x)$ is the maximal value of $f_\sigma(x)$ when σ runs through all values in I. That is

$$g(x) = \sup_{\sigma\in I}\{f_\sigma(x)\}.$$

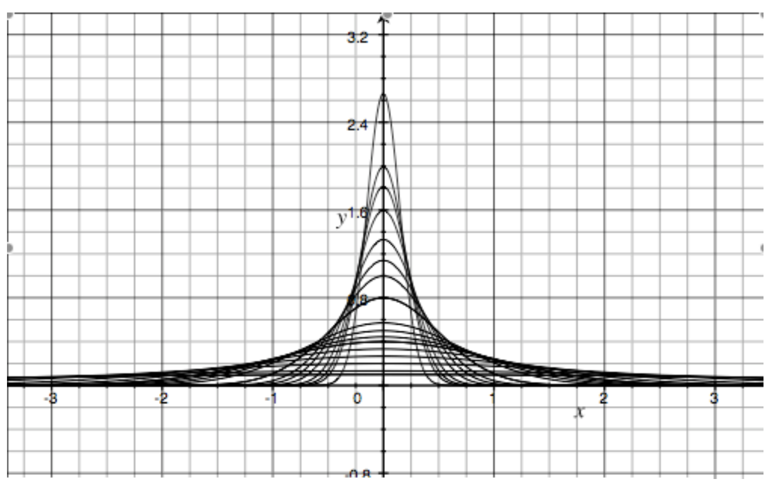
We say that g is the *upper envelope* of f_σ.

To work out the upper envelope of f_σ. We fix x and consider $f_\sigma(x)$ as a function of the variable σ. Then we seek its maximal value. If $I = [a, b]$ and $f_\sigma(x)$ is continuous in σ, then by the extreme value theorem, the maximal value of $f_\sigma(x)$ is attained at a point $\hat{\sigma}$. For different values of x, we may have a different maximal point $\hat{\sigma}$.

Example 7.31 Find the upper envelope for the family of Gaussian curves:

$$f(\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}}.$$

Graph for $\sigma = \{0.15, 0.2, 0.22, 0.25, 0.3, 0.35, 0.4, 0.5, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.4, 3, 4\}$.



Fix the value x. We compute $f(\sigma, x)$ for the critical value $\hat{\sigma}$ for which $f(\hat{\sigma}, x)$ is a maximum.

$$\partial f/\partial\sigma = \frac{1}{\sqrt{2\pi}}\left(-\frac{1}{\sigma^2} + \frac{x^2}{\sigma^4}\right)e^{\frac{-x^2}{2\sigma^2}}.$$

Set $\frac{\partial f}{\partial\sigma}(\sigma, x) = 0$ and obtain the solution $\sigma = |x|$.

Now

$$f''(\sigma, x) = \frac{1}{\sqrt{2\pi}}\left(\frac{2}{\sigma^3} - \frac{4x^2}{\sigma^5} - ((-\frac{1}{\sigma^2} + \frac{x^2}{\sigma^4})^2\right) e^{\frac{-x^2}{2\sigma^2}}.$$

Consequently

$$f''(\hat{\sigma}, x) = 2 - 4 - 1 + 1 < 0.$$

So $F(\sigma, x)$ achieves its maximum when $\sigma = x$. The curve $y = f(\hat{\sigma}, x)$ is now given by

$$y = \frac{1}{\sqrt{2\pi}|x|} e^{-\frac{1}{2}}.$$

It is called the *upper envelope* of the family of curves $f(\sigma)$. For each value of σ, the graph of f_σ touches the upper envelope at precisely $x = \sigma$.

Chapter 8

Higher order derivatives

8.1 Continuously Differentiable Functions

If f is differentiable at every point of an interval, we investigate the properties of its derivative $f'(x)$.

Definition 8.1 *We say that $f : [a, b] \to \mathbf{R}$ is continuously differentiable if it is differentiable on $[a, b]$ and f' is continuous on $[a, b]$. Instead of 'continuously differentiable' we also say that f is C^1 on $[a, b]$.*

Note that by $f'(a)$ we mean $f'(a+)$ and by $f'(b)$ we mean $f'(b-)$. A similar notion exists for f defined on (a, b), or on (a, ∞), on $(-\infty, b)$, or on $(-\infty, \infty)$.

Example 8.1

$$f(x) = \begin{cases} x^2 \sin(1/x) & x > 0 \\ 0, & x \neq 0 \end{cases}$$

Claim: The function f is differentiable. But f is not C^1.
Proof

$$f'(0) = \lim_{x \to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} x \sin(1/x) = 0.$$

So f is differentiable everywhere and

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & x > 0 \\ 0, & x \neq 0 \end{cases}$$

By the sandwich theorem, $\lim_{x \to 0} 2x \sin(1/x) = 0$. On the other hand $\lim_{x \to 0} \cos(1/x)$ does not exist. Why not? Thus $\lim_{x \to 0} f'(x)$ does not exist. $\square$

Example 8.2 Let $0 < \alpha < 1$ and

$$h(x) = \begin{cases} x^{2+\alpha}\sin(1/x) & x > 0 \\ 0, & x = 0 \end{cases}$$

Claim: The function h is C^1 on $[0, \infty)$.
Proof Since

$$h'(0) = \lim_{x\to 0} \frac{x^{2+\alpha}\sin(1/x)}{x - 0} = \lim_{x\to 0+} x^{1+\alpha}\sin(1/x) = 0,$$

$$h'(x) = \begin{cases} (2+\alpha)x^{1+\alpha}\sin(1/x) - x^\alpha\cos(1/x) & x > 0 \\ 0, & x = 0 \end{cases}$$

Since

$$|(2+\alpha)x^{1+\alpha}\sin(1/x) - x^\alpha\cos(1/x)| \le (2+\alpha)|x|^{1+\alpha} + |x|^\alpha,$$

by the sandwich theorem, $\lim_{x\to 0+} h'(x) = 0 = h'(0)$. Hence h is C^1. $\square$

Show that h' is not differentiable at $x = 0$, if $\alpha \in (0, 1)$.

Definition 8.2 *The set of all continuously differentiable functions on* $[a, b]$ *is denoted by* $C^1([a, b]; \mathbf{R})$ *or by* $C^1([a, b])$ *for short.*

Similarly we may define $C^1((a, b); \mathbf{R})$, $C^1((a, b]; \mathbf{R})$, $C^1([a, b); \mathbf{R})$, $C^1([a, \infty); \mathbf{R})$, $C^1((-\infty, b]; \mathbf{R})$, $C^1((-\infty, \infty); \mathbf{R})$ etc.

8.2 Higher Order Derivatives

Example 8.3 Let

$$g(x) = \begin{cases} x^3\sin(1/x) & x \ne 0 \\ 0, & x \ne 0 \end{cases}$$

Claim: The function g is twice differentiable on all of $\mathbf{R}$.
Proof t is clear that g is differentiable on $\mathbf{R} \setminus \{0\}$. Follow the proof of Example 8.1 to prove that g is differentiable at 0 and $g'(0) = 0$! So g is differentiable everywhere and its derivative is

$$g'(x) = \begin{cases} 3x^2\sin(1/x) - x\cos(1/x) & x > 0 \\ 0, & x \ne 0 \end{cases}$$

By the sandwich theorem, $\lim_{x\to 0} 3x^2\sin(1/x) - x\cos(1/x) = 0 = g'(0)$. Hence g is C^1.

$\square$

If the derivative of f is also smooth, we consider the derivative of f'.

Definition 8.3 *Let $f : (a, b) \to \mathbf{R}$ be differentiable. We say f is twice differentiable at x_0 if $f' : (a, b) \to \mathbf{R}$ is differentiable at x_0. We write:*

$$f''(x_0) = (f')'(x_0).$$

It is called the second order derivative of f at x_0. It is also denoted by $f^{(2)}(x_0)$.

Following from these

Definition 8.4 *We say that f is n times differentiable at x_0 if $f^{(n-1)}$ is differentiable at x_0. The the nth derivative of f at x_0 is:*

$$f^{(n)}(x_0) = (f^{(n-1)})'(x_0).$$

Definition 8.5 *We say that f is C^∞ on (a, b) if it is n times differentiable for every $n \in N$.*

Definition 8.6 *We say that f is n times continously differentiable on $[a, b]$ if $f^{(n)}$ exists and is continuous on $[a, b]$.*

Note that f is n times continosly differentiable on $[a, b]$ means that all the lower order derivatives and f itself are continuous.

Definition 8.7 *The set of all functions which are n times differentiable on (a, b) and such that the derivative $f^{(0)}, f^{(1)}, \dots f^{(n)}$ are continuous functions on (a, b) is denoted by $C^n(a, b)$.*

There is a similar definition with $[a, b]$, or other types of intervals, in place of (a, b).

8.3 Convexity

To finish we give a theorem for deciding whether a critical point is a local minimum or a local maximum.

Consider the two functions: $f(x) = x^2$ and $g(x) = 1 - x^2$ with critical point $x = 0$. We have $f'' > 0$ and $g'' < 0$. What do their graphs look like ?

Theorem 8.1 *Consider a differentiable function $f : (a, b) \to \mathbf{R}$. Suppose that $f'(x_0) = 0$. Then f has a local minimum at x_0 if $f''(x_0) > 0$; f has a local maximum if $f''(x_0) < 0$.*

Proof Suppose that $f''(x_0) > 0$ then for some $r > 0$, $f'(x) > f'(x_0) = 0$ on $(x_0, x_0 + r)$ and $f'(x) < f'(x_0) = 0$ on $(x_0 - r, x_0)$. Since $f'(x_0) = 0$, $f'(x) > 0$ for $x \in (x_0, x_0 + r)$ and $f'(x) < 0$ for $x \in (x_0 - r, x_0)$.

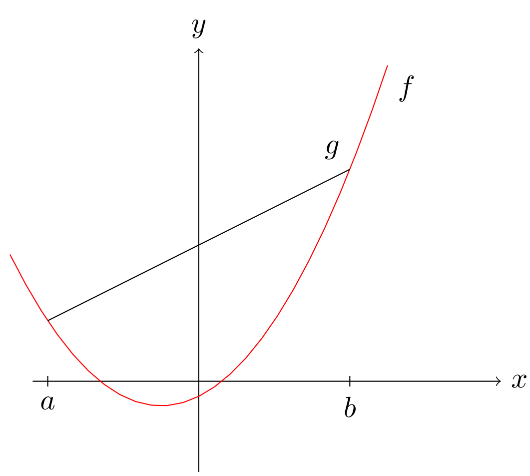
It follows that f is increasing on $(x_0, x_0 + r)$ and f is decreasing on $(x_0 - r, x_0)$. Thus $f(x_0)$ is a local minimum.

The case where $f''(x_0) < 0$ follows from this by taking $g = -f$. □

Remark 8.1 If f is differentiable on (a, b) and $f'(x_0) = f''(x_0) = 0$, what can we say about the maximality/minimality of f at x_0? e.g. case $f^{(3)}(x_0) > 0$? case $f^{(3)}(x_0) < 0$? Construct representative examples! In the first case show that $f'(x) > 0$ on a neighbourhood of x_0 and on which f is increasing. Discuss the case when $f'(x_0) = f''(x_0) = \cdots = f^{(n)}(x_0) = 0$ and $f^{(n+1)}(x_0) \neq 0$.

Corollary 8.2 *If $f : (a, b) \to \mathbf{R}$ is such that $f''(x) > 0$ for all $x \in (a, b)$ and f is continuous on $[a, b]$ then for all $x \in (a, b)$,*

$$f(x) \leq f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$



Proof Let

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

The graph of g is the straight line joining the points $(a, f(a))$ and $(b, f(b))$. Let $h(x) = f(x) - g(x)$. Then

1. $h(a) = h(b) = 0$;

2. h is continuous on $[a, b]$ and twice differentiable on (a, b);

3. $h''(x) = f''(x) - g''(x) = f''(x) > 0$ for all $x \in (a, b)$.

By the extreme value theorem there are points $x_1, x_2 \in [a, b]$ such that

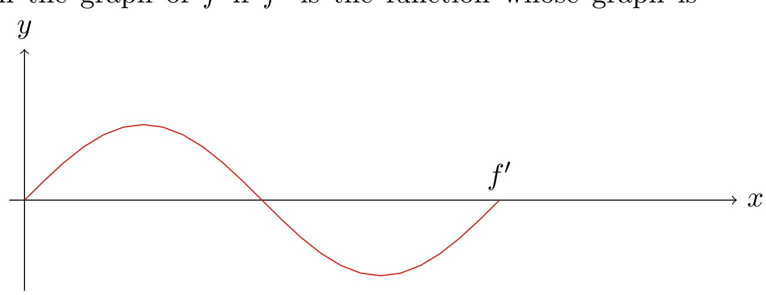
$$h(x_1) \le h(x) \le h(x_2), \qquad \forall x \in [a, b].$$

If $x_2 \in (a, b)$ it would be a local maximum and hence $f'(x_2) = 0$ (see Lemma 7.1). By Theorem 8.1, x_2 would have to be the local minimum since $h''(x_2) = f''(x_2) > 0$. So x_2 is a or b and in either case $h(x_2) = 0$. We have proved that 0 is the maximum value of h and so $h(x) \le 0$ and $f(x) \le g(x)$. □

Remark 8.2 1. Let f be a function for which $f''(x) > 0$ for all x. Then the region above the graph of f is convex i.e. for any two points in this region, the line segment joining them is completely contained in the region. For this reason, such functions are called *convex functions*.

2. Any point $x \in [a, b]$ can be written as $(1 - t)a + tb$, $t \in [0, 1]$.

Exercise 8.4 Sketch the graph of f if f' is the function whose graph is



shown and $f(0) = 0$.

Chapter 9

Power Series Functions

Definition 9.1 *By a power series we mean the formal expression:*

$$\sum_{n=0}^{\infty} a_n x^n$$

where each a_n is in $\mathbf{R}$ *(or, later on in the subject, in* $\mathbf{C}$*).*

By convention, $x^0 = 1$ and $0^n = 0$.

Example 9.1 1. $\sum_{n=0}^{\infty} (\frac{x}{3})^n$, $\qquad a_n = \frac{1}{3^n}, n = 0, 1, 2 \dots$.

2. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$, $\qquad a_0 = 0,\ a_n = (-1)^n \frac{1}{n},\ n = 1, 2, \dots$.

3. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\qquad a_n = \frac{1}{n!},\ n = 0, 1, 2, \dots$. By convention $0! = 1$.

4. $\sum_{n=0}^{\infty} n! x^n$, $\qquad a_n = n!, n = 0, 1, 2, \dots$.

If we substitute a real number in place of x, we obtain an infinite series. For which values of x does the infinite series converge?

Lemma 9.1 (The Ratio test) *The series* $\sum_{n=0}^{\infty} b_n$

1. *converges absolutely if there is a number $r < 1$ such that $\left|\frac{b_{n+1}}{b_n}\right| < r$ eventually.*

2. *The series $\sum_{n=0}^{\infty} b_n$ diverges if there is a number $a > 1$ such that $\left|\frac{b_{n+1}}{b_n}\right| > a$, eventually.*

Recall by a statement holds 'eventually' we mean that there exists N_0 such that the statement holds for all $n > N_0$.

Lemma 9.2 (The Limiting Ratio test) *The series* $\sum_{n=0}^{\infty} b_n$

1. *converges absolutely if* $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| < 1$;

2. *The series* $\sum_{n=0}^{\infty} b_n$ *diverges if* $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| > 1$.

Example 9.2 Consider $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$. Let $b_n = \left(\frac{x}{3}\right)^n$. Then

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{\left|(\frac{x}{3})^{n+1}\right|}{\left|(\frac{x}{3})^n\right|} = \frac{|x|}{3}, \qquad \begin{cases} < 1, & \text{if } |x| < 3 \\ 1, & \text{if } |x| > 3. \end{cases}$$

By the ratio test, the power series $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$ is absolutely convergent if $x \in (-3, 3)$. The power series $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$ diverges if $|x| > 3$.
If $x = 3$, the power series, $\sum_{n=0}^{\infty} \left(\frac{3}{3}\right)^n = 1 + 1 + \dots$, is divergent.
If $x = -3$ the power series, $(-1) + 1 + (-1) + 1 + \dots$, is divergent.

Example 9.3 Consider $\sum_{n=0}^{\infty} \frac{(-1)^n}{n} x^n$. Let $b_n = \frac{(-1)^n}{n} x^n$. Then

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{\left|\frac{(-1)^{n+1}}{n+1} x^{n+1}\right|}{\left|\frac{(-1)^n}{n} x^n\right|} = |x| \left(\frac{n}{n+1}\right) \xrightarrow{n\to\infty} |x|, \qquad \begin{cases} < 1, & \text{if } |x| < 1 \\ 1, & \text{if } |x| > 1. \end{cases}$$

For $x \in (-1, 1)$, the power series $\sum \frac{(-1)^n}{n} x^n$ is convergent.
It is divergent if $|x| > 1$.
For $x = 1$, we have $\sum \frac{(-1)^n}{n}$ which is convergent.
For $x = -1$ we have $\sum \frac{1}{n}$ which is divergent.

Example 9.4 Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. We have

$$\left|\frac{\frac{x^{(n+1)}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \frac{|x|}{n+1} \to 0 < 1$$

The power series converges for all $x \in (-\infty, \infty)$.
$\sum_{n=0}^{\infty} n! x^n$ converges only at $x = 0$. Indeed if $x \neq 0$,

$$\frac{|(n+1)x^{(n+1)|}}{|n!x^n|} = (n+1)|x| \to \infty$$

as $n \to \infty$.

If for some $x \in \mathbf{R}$, the infinite series $\sum_{n=0}^{\infty} a_n x^n$ converges, we define

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

The domain of f consists of all x such that $\sum_{n=1}^{\infty} a_n x^n$ is convergent. If $x = 0$,

$$f(0) = \sum_{n=0}^{\infty} a_n 0^n = 0$$

is convergent. We wish to identify the set of points for which the power series is convergent. Also, is the function so defined continuous? differentiable?

9.1 Radius of Convergence

Lemma 9.3 *If for some number x_0, $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then $\sum_{n=0}^{\infty} a_n C^n$ converges absolutely for any C with $|C| < |x_0|$.*

Proof As $\sum a_n x_0^n$ is convergent,

$$\lim_{n\to\infty} |a_n||x_0|^n = 0.$$

Convergent sequences are bounded. So there is a number M such that for all n, $|a_n x_0^n| \le M$.

Suppose $|C| < |x_0|$. Let $r = \frac{|C|}{|x_0|}$. Then $r < 1$, and

$$|a_n C^n| = \left| a_n x_0^n \left(\frac{C^n}{x_0^n} \right) \right| \le M r^n.$$

By the comparison theorem, $\sum_{n=0}^{\infty} a_n C^n$ converges absolutely. $\square$

Theorem 9.4 *Consider a power series $\sum_{n=0}^{\infty} a_n x^n$. Then one of the following holds:*

(1) The series only converges at $x = 0$.

(2) The series converges for all $x \in (-\infty, \infty)$.

(3) There is a positive number $0 < R < \infty$, such that

- *$\sum_{n=0}^{\infty} a_n x^n$ converges for all x with $|x| < R$;*
- *$\sum_{n=0}^{\infty} a_n x^n$ diverges for all x with $|x| > R$.*

This number R is called the radius of convergence. In case (1) we define the radius of convergence to be 0, and in case (2) we define it to be ∞.

Proof

- Let
$$S = \left\{ x \;\Big|\; \sum_{n=0}^{\infty} a_n x^n \text{ is convergent} \right\}.$$
Since $0 \in S$, S is not empty. If S is not bounded above, then by Lemma 9.3, $\sum_{n=0}^{\infty} a_n x^n$ converges for all x. This is case (2).

- Either there is a point $y_0 \neq 0$ such that $\sum_{n=0}^{\infty} a_n y_0^n$ is convergent, or $R = 0$ and we are in case (1).

- If S is strictly bigger than just $\{0\}$, and is bounded, let
$$R = \sup\{|x| : x \in S\}.$$
  - If x is such that $|x| < R$, then $|x|$ is *not* an upper bound for S (after all, R is the *least* upper bound). Thus there is a number $b \in S$ with $|x| < b$. Because $\sum_{n=0}^{\infty} a_n b^n$ converges, by Lemma 9.3 it follows that $\sum_{n=0}^{\infty} a_n x^n$ converges.
  - If $|x| > R$, then $x \notin S$ hence $\sum_{n=0}^{\infty} a_n x^n$ diverges.

□

When $x = -R$ and $x = R$, we cannot make any conclusion about the convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ without further examination.

Lemma 9.5 *Algebraic operations on power series. Suppose that $\sum_{i=0}^{\infty} a_i x^i$ has radius of convergence R and $\sum_{i=0}^{\infty} b_i x^i$ has radius of convergence S. Then*
$$\sum_{i=0}^{\infty} (a_i + b_i) x^i \quad \textit{and} \quad \sum_{i=0}^{\infty} c a_i x^i$$
have radius of convergence $\min(R, S)$ and R respectively.

9.2 The Limit Superior of a Sequence

Recall that if A is a set then
$$\sup A = \begin{cases} \text{the least upper bound of } A, & \text{if } A \text{ is bounded above} \\ +\infty, & \text{if } A \text{ is not bounded above} \end{cases}$$

$$\inf A = \begin{cases} \text{the greatest lower bound of } A, & \text{if } A \text{ is bounded below} \\ -\infty, & \text{if } A \text{ is not bounded below} \end{cases}$$

Definition 9.2 *We define the limit superior of a sequence $\{b_n\}$, denoted by $\limsup_{n\to\infty} b_n$, to be*

$$\lim_{n\to\infty} \sup_{k\geq n} b_k.$$

For each n, $A_n := \{b_n, b_{n+1}, \dots\}$ is a set. We write $\sup_{k\geq n} b_k = \sup A_n$ for short. Note that $a_n := \sup A_n$ is non-increasing in n, for $A_n \subset A_{n-1}$. By the monotone convergence theorem $\lim_{n\to\infty} a_n$ is well defined; it is a finite number if $\{a_n : n = 1, 2, \dots\}$ is bounded below and it equals $-\infty$ if $\{a_n : n = 1, 2, \dots\}$ is not bounded below.

Example 9.5

1. Let $b_n = 1 - \frac{1}{n}$. Then

$$\sup\left\{1 - \frac{1}{n}, 1 - \frac{1}{n+1}, \dots\right\} = 1.$$

Hence

$$\limsup_{n\to\infty}(1 - \frac{1}{n}) = \lim_{n\to\infty} 1 = 1.$$

2. Let $b_n = 1 + \frac{1}{n}$. Then $\sup\{1 + \frac{1}{n}, 1 + \frac{1}{n+1}, \dots\} = 1 + \frac{1}{n}$. Hence

$$\limsup_{n\to\infty}(1+\frac{1}{n}) = \lim_{n\to\infty} \sup\left\{1 + \frac{1}{n}, 1 + \frac{1}{n+1}, \dots\right\} = \lim_{n\to\infty}(1+\frac{1}{n}) = 1.$$

Recall from Analysis I that

Lemma 9.6 *If $a_1 \geq a_2 \geq a_3 \dots$ is a non-increasing sequence, then $\inf_{n\geq 1} a_n = \lim_{n\to\infty} a_n$.*

Remark 9.1 The limit superior of the sequence $\{b_n\}$ always exists and

$$\limsup_{n\to\infty} b_n = \inf_{n\geq 1} \sup_{k\geq n} b_k.$$

Definition 9.3 *The limit inferior of $\{b_n\}$ is defined as:*

$$\liminf_{n\to\infty} b_n = \sup_{n\geq 1} \inf_{k\geq n} b_k.$$

Proposition 9.7 *If $l = \limsup_{n\to\infty} b_n$ is a finite number, then for all $\epsilon > 0$,*

1. *There is a natural number N_ϵ such that*

$$b_n < l + \epsilon, \ \forall n > N_\epsilon.$$

2. *For any natural number N,*

$$\textit{there exists } n > N \textit{ s.t.} \quad b_n > l - \epsilon$$

The moral of this lemma is this: for any $\epsilon > 0$, b_n is bounded above by $l + \epsilon$ eventually. For any $\epsilon > 0$, there is a sub-sequence b_{n_k} with $b_{n_k} > l - \epsilon$.

Proof Recall that $l = \inf_{n\geq 1} \sup_{k\geq n} b_k$.

1. By definition of the *greatest* lower bound, for any $\epsilon > 0$ there is N such that
$$\sup_{k\geq N} b_k \leq l + \epsilon.$$
Hence $b_k \leq l + \epsilon$ for all $n \geq N$, proving part 1.

2. Since l is a lower bound of $\sup_{k\geq n} b_k$,
$$\sup_{k\geq n} b_k \geq l, \qquad \forall n.$$
Let N be any natural number. By definition of the *least* upper bound $\sup_{k\geq N} b_k$, for any $\epsilon > 0$, there is $n \geq N$ such that
$$b_n \geq \sup_{k\geq N} b_k - \epsilon \geq l - \epsilon$$
giving part 2.

$\square$

Corollary 9.8 *If $l = \limsup_{n\to\infty} b_n$ there is a sub-sequence b_{n_k} such that $\lim_{k\to\infty} b_{n_k} = l$.*

Proof Let k be any natural number and take $\epsilon = \frac{1}{k}$ in Proposition 9.7. Then by part 2 of Proposition 9.7 there is $n_k \geq k$ s.t. $b_{n_k} > l - \frac{1}{k}$.

Then $\lim_{k\to\infty} b_{n_k} = l$. $\square$

Definition 9.4 *If a is a number such that there is a sub-sequence $\{b_{n_k}\}$ with*

$$\lim_{k\to\infty} b_{n_k} = a,$$

we say a is a limit point of $\{b_n\}$.

Theorem 9.9 *Let S be the set of limit points of $\{b_n\}$. Then*

$$\limsup_{n\to\infty} b_n = \max S;$$

$$\liminf_{n\to\infty} b_n = \min S.$$

Theorem 9.10 *If $\lim_{n\to\infty} b_n = b > 0$ then*

$$\limsup_{n\to\infty}(b_n a_n) = b \limsup_{n\to\infty} a_n.$$

Proof [Proof 1]. For any $0 < \epsilon < b/2$ there exists N such that if $n > N$,

$$b - \epsilon < b_n < b + \epsilon. \tag{9.1}$$

By definition

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \dots\}.$$

Suppose first that $\limsup_{n\to\infty} a_n \geq 0$. If $n > N$, then by (9.1),

$$(b-\epsilon)\sup\{a_n, a_{n+1}, \dots\} \leq \sup\{a_n b_n, a_{n+1}b_{n+1}, \dots\} < (b+\epsilon)\sup\{a_n, a_{n+1}, \dots\}. \tag{9.2}$$

If $\limsup_{n\to\infty} a_n \leq 0$, these inequalities are reversed.

Taking the limit as n goes to ∞ of all three terms in (9.2),

$$(b-\epsilon)\lim_{n\to\infty}\sup\{a_n, a_{n+1}, \dots\} \leq \lim_{n\to\infty}\sup\{a_n b_n, a_{n+1}b_{n+1}, \dots\} \leq (b+\epsilon)\lim_{n\to\infty}\sup\{a_n, a_{n+1}, \dots\}.$$

which means for all $\epsilon \in (0, b/2)$,

$$(b-\epsilon)\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n b_n) \leq (b+\epsilon)\limsup_{n\to\infty} a_n.$$

As this holds for all $\epsilon > 0$, it follows that

$$b\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n b_n) \leq b\limsup_{n\to\infty} a_n.$$

□

Proof [Proof 2]

1. The limit points of $\{a_n b_n\}$ are limit points of $\{a_n\}$ multiplied by b.

   To prove this, note the following. If there is a subsequence of $a_n b_n$ such that
$$\lim_{k\to\infty} a_{n_k} b_{n_k} = \ell,$$
   then
$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} \frac{a_{n_k} b_{n_k}}{b_{n_k}} = \ell/b.$$
   If $\lim_{k\to\infty} a_{n_k} = c$, then
$$\lim_{k\to\infty} a_{n_k} b_{n_k} = cb,$$

2. Let
$$S = \{x : \ x \text{ is a limit point of } \{a_n\}\}.$$
   If $b > 0$, then multiplying the members of the set S by b preserves their order. Since
$$\limsup_{nto\infty}(a_n b_n) = \max\{bx : x \in S\}$$
   by Step 1, and
$$\max\{bx : x \in S\} = b \max S = b \limsup_{n\to\infty} a_n,$$
   the result follows.

□

9.3 Hadamard's Test

Theorem 9.11 (Cauchy's root test) *The infinite sum* $\sum_{n=1}^{\infty} z_n$ *is*

1. *convergent if* $\limsup_{n\to\infty} |z_n|^{\frac{1}{n}} < 1$.
2. *divergent if* $\limsup_{n\to\infty} |z_n|^{\frac{1}{n}} > 1$.

Proof Let
$$a = \limsup_{n\to\infty} |z_n|^{\frac{1}{n}}.$$

1. Suppose that $a < 1$. Choose $\epsilon > 0$ so that $a+\epsilon < 1$. Then there exists N_0 such that whenever $n > N_0$, $|z_n| < (a+\epsilon)^n$. As $\sum_{n=1}^{\infty}(a+\epsilon)^n$ is convergent, $\sum_{n=N_0}^{\infty}|z_n|$ is convergent by comparison. And
$$\sum_{n=1}^{\infty}|z_n| = \sum_{n=1}^{N_0-1}|z_n| + \sum_{n=N_0}^{\infty}|z_n|$$
is convergent.

2. Suppose that $a > 1$. Choose ϵ so that $a-\epsilon > 1$. Then for infinitely many n, $|z_n| > (a-\epsilon)^n > 1$. Consequently $|z_n| \not\to 0$ and hence $\sum_{n=1}^{\infty} z_n$ is divergent.

□

Theorem 9.12 *[Hadamard's Theorem/Hadamard's Test] Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series.*

1. *If* $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \infty$, *the power series converges only at* $x = 0$.
2. *If* $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$, *the power series converges for all* $x \in (-\infty, \infty)$.
3. *If* $0 < r = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < \infty$, *the radius of convergence* R *of the power series is given by*
$$R = \frac{1}{r}.$$

Proof We only prove case 3. If $|x| < R$, $\limsup_{n\to\infty} |a_n x^n|^{\frac{1}{n}} = r|x| < 1$ and by Cauchy's root test the series $\sum_{n=0}^{\infty} a_n x^n$ converges.

Suppose that $|x| > R$, then $\limsup_{n\to\infty} |a_n x^n|^{\frac{1}{n}} = r|x| > 1$, and by Cauchy's root test, the series $\sum_{n=0}^{\infty} a_n x^n$ diverges.

□

Example 9.6 Consider $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$. We have
$$\lim_{n\to\infty}\left(\frac{|(-3)^n|}{\sqrt{n+1}}\right)^{\frac{1}{n}} = 3\lim_{n\to\infty}\frac{1}{(n+1)^{\frac{1}{2n}}} = 3.$$
So $R = \frac{1}{3}$. If $x = -1/3$, we have a divergent sequence. The interval of convergence is $x = (-1/3, 1/3]$.

Example 9.7 Consider $\sum_{n=1}^{\infty} a_n x^n$ where $a_{2k} = 2^{2k}$ and $a_{2k+1} = 5(3^{2k+1})$. This means

$$a_n = \begin{cases} 2^n, & n \text{ is even} \\ 5(3^n), & n \text{ is odd} \end{cases}$$

Thus

$$|a_n|^{\frac{1}{n}} = \begin{cases} 2, & n \text{ is even} \\ 3(5^{\frac{1}{n}}), & n \text{ is odd} \end{cases},$$

which has two limit points: 2 and 3.

$$\limsup |a_n|^{\frac{1}{n}} = 3.$$

and $R = 1/3$. For $x = \frac{1}{3}$, $x = -\frac{1}{3}$, $|a_n x^n| \neq 0$ and does not converge to 0 (check). So the radius of convergence is $(-1/3, 1/3)$.

Example 9.8 Consider $\sum_{n=1}^{\infty} a_n x^n$ where $a_{2k} = (\frac{1}{5})^{2k}$ and $a_{2k+1} = \left(-\frac{1}{3}\right)^{2k+1}$.

$$\limsup |a_n|^{\frac{1}{n}} = 1/3.$$

So $R = 3$. For $x = 3$, $x = -3$, $|a_n x^n| \geq 1$ and does not tend to 0. So the interval of convergence is $(-3, 3)$.

Example 9.9 Consider the series $\sum_{k=1}^{\infty} k^2 x^{k^2}$. We have

$$a_n = \begin{cases} n, & \text{if } n \text{ is the square of a natural number} \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\limsup |a_n|^{\frac{1}{n}} = 1$ and $R = 1$. The interval of convergence is $(-1, 1)$.

9.4 Continuity of Power Series Functions

If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, define $f : (-R, R) \to \mathbf{R}$ by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Let $f_N(x) = \sum_{n=0}^{N} a_n x^n$; then for each $x \in (-R, R)$,

$$f(x) = \lim_{N \to \infty} f_N(x).$$

Each function f_N is continuous and differentiable; can we say the same about f? In Analysis 3 we'll introduce uniform convergence and show that the limit of a sequence of continuous functions is continuous provided that the convergence is uniform.

In this section we concentrate on the limit of polynomial functions. Let us first consider continuity.

Proposition 9.13 *Let $f : (-R, R) \to \mathbf{R}$ by*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where R is the radius of convergence of the power series. Then f is continuous at every x in $(-R, R)$.

Proof Fix $x_0 \in (-R, R)$. Choose r such that $[x_0 - r, x_0 + r] \subset (-R, R)$. Let $\rho = \max\{|x_0 - r|, |x_0 + r|\}$ Note that $\rho < R$.

For $\epsilon > 0$, there exists N such that if $n > N$, the tail of the convergent series $\sum_{n=1}^{\infty} |a_n|\rho^n$ satisfies:

$$\sum_{n=N+1}^{\infty} |a_n|\rho^n < (1/3)\epsilon$$

Consequently

$$\sum_{n=N+1}^{\infty} |a_n||x^n| \le \sum_{n=N+1}^{\infty} |a_n|\rho^n < (1/3)\epsilon \qquad \forall x \in [x_0 - r, x_0 + r].$$

It follows that

$$\begin{aligned}
|f(x) - f(x_0)| &= \left|\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x_0^n\right| \\
&\le |\sum_{n=0}^{N} a_n x^n - \sum_{n=0}^{N} a_n x_0^n| + \sum_{n=N+1}^{\infty} |a_n x^n| + \sum_{n=N+1}^{\infty} |a_n x_0^n| \\
&\le \sum_{n=0}^{N} a_n |x^n - x_0^n| + \epsilon/3 + \epsilon/3
\end{aligned}$$

Let $A = \sum_{n=0}^{N} |a_n|$. Since x^n is continuous for all n, for each $n < N$ we can choose $\delta_n > 0$ such that if $|x - x_0| < \delta_n$,

$$|x^n - x_0^n| < \frac{\epsilon}{3A}.$$

Let $\delta = \min\{\delta_1, \ldots, \delta_N, r\}$, Then if $|x - x_0| < \delta$,

$$\sum_{n=0}^{N} a_n |x^n - x_0^n| \le \sum_{n=0}^{N} a_n \left(\frac{\epsilon}{3A}\right) \le \left(\frac{\epsilon}{3A}\right) \sum_{n=0}^{N} a_n = \frac{\epsilon}{3}.$$

The required continuity now follows since if $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| \le \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

□

9.5 Term by Term Differentiation

If $f_N(x) = \sum_{n=0}^{N} a_n x^n = a_0 + a_1 x + \cdots + a_N x^N$ then

$$f_N'(x) = \sum_{n=1}^{N} n a_n x^{n-1}, \qquad f_N''(x) = \sum_{n=2}^{N} n(n-1) a_n x^{n-2}.$$

Does it hold, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, that

$$\begin{aligned} f'(x) &\equiv \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=1}^{\infty} n a_n x^{n-1} \\ f''(x) &\equiv \frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad ? \end{aligned}$$

These equalities hold if it is true that the derivative of a power series is the sum of the derivatives of its terms; in other words, we are asking whether it is correct to differentiate a power series *term by term*. As preliminary question, we ask: if R is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$, what are the radii of convergence of the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$?

Lemma 9.14 *The power series $\sum_{n=0}^{\infty} a_n x^n$, $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ have the same radius of convergence.*

Proof Let R_1, R_2, R_3 be respectively the radii of convergence of the first, the second and the third series. Recall Hadamard's Theorem: if $\ell = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$, then $R_1 = \frac{1}{\ell}$. We interpret $\frac{1}{0}$ as ∞ and $\frac{1}{\infty}$ as 0 to cover all three cases for the radius of convergence.

Observe that

$$x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = \sum_{n=1}^{\infty} n a_n x^n$$

so $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=1}^{\infty} n a_n x^n$ have the same radius of convergence R_2. By Hadamard's Theorem

$$R_2 = \frac{1}{\limsup_{n\to\infty} |n a_n|^{\frac{1}{n}}}$$

and note that

$$\limsup_{n\to\infty} |n a_n|^{\frac{1}{n}} = \lim_{n\to\infty} n^{\frac{1}{n}} \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \ell.$$

The last step follows from the fact that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ and from Theorem 9.9. So $R_1 = R_2$.

□ Exercise: show that $R_3 = R_1$. You may use the fact that

$$\lim_{n\to\infty} (n-1)^{\frac{1}{n}} = 1.$$

Below we prove term by term differentiation, and in the process show that f is differentiable on $(-R, R)$. From this theorem it follows that f is continuous. This means that strictly speaking we did not need the separate proof of continuity that we gave in Theorem 9.13. We gave this other proof because the idea of the proof will be used again in Analysis III, and therefore it is useful to become familiar with it.

Theorem 9.15 *[Term by Term Differentiation] Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. Let $f : (-R, R) \to \mathbf{R}$ be the function defined by this power series,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then f is differentiable at every point of $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Proof Let $x_0 \in (-R, R)$. We show that f is differentiable at x_0 and $f'(x_0) = \sum_{n=1}^{\infty} n a_n x_0^{n-1}$. We wish to show that for any $\epsilon > 0$ there is $\delta > 0$ such that if $|x - x_0| < \delta$,

$$\left| \frac{|f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \epsilon. \tag{9.3}$$

We simplify the left hand side:

$$\begin{aligned}
& \left| \frac{|f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \\
= & \left| \frac{\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x_0^n}{x - x_0} - \sum_{n=1}^{\infty} n a_n x_0^{n-1} \right| \\
= & \left| \sum_{n=1}^{\infty} \frac{a_n (x^n - x_0^n)}{x - x_0} - \sum_{n=1}^{\infty} n a_n x_0^{n-1} \right| \\
= & \left| \sum_{n=2}^{\infty} a_n \left[\frac{x^n - x_0^n}{x - x_0} - n x_0^{n-1} \right] \right|
\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} |a_n| \left| \frac{x^n - x_0^n}{x - x_0} - n x_0^{n-1} \right|. \tag{9.4}$$

To prove (9.3), we thus need to control each term $\left| \frac{x^n - x_0^n}{x - x_0} - n x_0^{n-1} \right|$. We do this by means of the following lemma.

Note that if $x \in (a, b)$ then $|x| \leq \max(a, b)$; similarly

$$\max\{|x| : x \in [x_0 - \delta, x_0 + \delta]\} = \max\{|x_0 - \delta|, |x_0 + \delta|\}.$$

Lemma 9.16 *Let $x_0 \in \mathbf{R}$ and let $n \in \mathbf{N}$. If $\delta > 0$ and $0 < |x - x_0| < \delta$, then*

$$\left| \frac{x^n - x_0^n}{x - x_0} - n x_0^{n-1} \right| \leq n(n-1)\rho^{n-2} |x - x_0|,$$

where $\rho = \max(|x_0 - \delta|, |x_0 + \delta|)$.

Proof We assume that $x > x_0$. The case that $x < x_0$ can be proved analogously. Apply the Mean Value Theorem to $f_n(x) = x^n$, on $[x_0, x]$: there exists $c_n \in (x_0, x)$ such that

$$\frac{f_n(x) - f_n(x_0)}{x - x_0} = f_n'(c_n).$$

Since $f_n'(x) = nx^{n-1}$ this becomes

$$\frac{x^n - x_0^n}{x - x_0} = nc_n^{n-1}.$$

It follows that

$$\begin{aligned}\left|\frac{x^n - x_0^n}{x - x_0} - nx_0^{n-1}\right| &= |nc_n^{n-1} - nx_0^{n-1}| \\ &= n|c_n^{n-1} - x_0^{n-1}|.\end{aligned}$$

Apply the mean value theorem again, this time to $f_{n-1}(x) = x^{n-1}$ on $[x_0, c_n]$: there exists $\xi_n \in (x_0, c_n)$ such that

$$\frac{f_{n-1}(x) - f_{n-1}(x_0)}{x - x_0} = f_{n-1}'(\xi_n).$$

Since $f_{n-1}'(x) = \frac{d}{dx}(x^{n-1}) = (n-1)x^{n-2}$,

$$\frac{c_n^{n-1} - x_0^{n-1}}{c_n - x_0} = (n-1)(\xi_n)^{n-2}$$

and

$$|c_n^{n-1} - x_0^{n-1}| = (n-1)|\xi_n|^{n-2}|c_n - x_0|.$$

We see that

$$\begin{aligned}\left|\frac{x^n - x_0^n}{x - x_0} - nx_0^{n-1}\right| &= n|c_n^{n-1} - x_0^{n-1}| \\ &= n(n-1)|\xi_n|^{n-2}|c_n - x_0|.\end{aligned}$$

Since $\xi_n \in (x_0, c_n) \subset (x_0, x)$, $|\xi_n| \leq \rho$ and $|c_n - x_0| \leq |x - x_0|$ we finally obtain that

$$\left|\frac{x^n - x_0^n}{x - x_0} - nx_0^{n-1}\right| \leq n(n-1)\rho^{n-2}|x - x_0|.$$

□

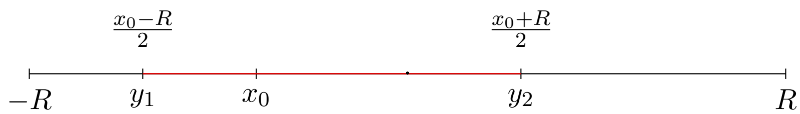
Proof of 9.15 (continued): Applying the Lemma to each of the terms in (9.4), we get

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| \leq \left[\sum_{n=2}^{\infty} n(n-1)|a_n|\rho^{n-2}\right] |x - x_0|.$$

We need to chose δ so that if $\rho = \max\{|x| : x \in [x_0 - \delta, x_0 + \delta]\}$, then the power series $\sum_{n=2}^{\infty} n(n-1)|a_n|\rho^{n-2}$ converges. Since, by Lemma 9.14, this power series has radius of convergence R, it converges if $\rho < R$.

We must now choose δ. We use the midpoint y_1 of $[-R, x_0]$ and the midpoint y_2 of $[x_0, R]$. We then choose a sub-interval

$$(x_0 - \delta_0, x_0 + \delta_0) \subset (y_1, y_2).$$



Let

$$\delta_0 := \min\{\frac{1}{2}|R - x_0|, \frac{1}{2}|x_0 + R|\}$$

and

$$\rho := \max\{\frac{1}{2}|R - x_0|, \frac{1}{2}|x_0 + R|\}.$$

Then $\rho < R$, as required.

Since the radius of convergence of $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ is also R, and $\rho < R$,

$$\sum_{n=2}^{\infty} n(n-1)a_n\rho^{n-2}$$

is absolutely convergent. Let

$$A := \sum_{n=2}^{\infty} n(n-1)|a_n|\rho^{n-2} < \infty.$$

Finally, let $\delta = \min\{\frac{\epsilon}{A}, \delta_0\}$. Then if $0 < |x - x_0| < \delta$,

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < A|x - x_0| \le \epsilon.$$

$\square$

Since the derivative of f is a power series with the same radius of convergence, we apply Theorem 9.15 to f' to see that f is twice differentiable with the second derivative again a power series with the same radius of convergence. Iterating this procedure we obtain the following corollary.

Corollary 9.17 *Let $f : (-R, R) \to \mathbf{R}$,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where R is the radius of convergence of the power series. Then f is in $C^\infty(-R, R)$.

Example 9.10 If $x \in (-1, 1)$, evaluate $\sum_{n=0}^{\infty} nx^n$.

Solution. The power series $\sum_{n=0}^{\infty} x^n$ has $R = 1$. But if $|x| < 1$ the geometric series has a limit and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

By term by term differentiation, for $x \in (-1, 1)$,

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} nx^{n-1}.$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} nx^n &= x\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) \\ &= x\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}. \end{aligned}$$

Chapter 10

Classical Functions of Analysis

The following power series have radius of convergence R equal to ∞.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

You can check this yourself using Theorem 9.12.

Definition 10.1 1. *We define the exponential function* $\exp : \mathbf{R} \to \mathbf{R}$ *by*

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \qquad \forall x \in \mathbf{R}.$$

2. *We define the sine function* $\sin : \mathbf{R} \to \mathbf{R}$ *by*

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \end{aligned}$$

3. *We define the cosine function* $\cos : \mathbf{R} \to \mathbf{R}$ *by*

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \end{aligned}$$

10.1 The Exponential and the Natural Logarithm Function

Consider the exponential function $\exp : \mathbf{R} \to \mathbf{R}$,

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \qquad \forall x \in \mathbf{R}.$$

Note that $\exp(0) = 1$. By the term-by-term differentiaiton theorem 9.15, $\frac{d}{dx}\exp(x) = \exp(x)$, and so exp is infinitely differentiable.

Proposition 10.1 *For all $x, y \in \mathbf{R}$,*

$$\begin{aligned} \exp(x+y) &= \exp(x)\exp(y), \\ \exp(-x) &= \frac{1}{\exp(x)}. \end{aligned}$$

Proof For any $y \in \mathbf{R}$ considered as a fixed number, let

$$f(x) = \exp(x+y)\exp(-x).$$

Then

$$\begin{aligned} f'(x) &= \frac{d}{dx}\exp(x+y)\exp(-x) + \exp(x+y)\frac{d}{dx}\exp(-x) \\ &= \exp(x+y)\exp(-x) + \exp(x+y)[-\exp(-x)] \\ &= 0. \end{aligned}$$

By the corollary to the Mean Value Theorem, $f(x)$ is a constant. Since $f(0) = \exp(y)$, $f(x) = \exp(y)$, i.e.

$$\exp(x+y)\exp(-x) = \exp(y).$$

Take $y = 0$, we have

$$\exp(x+0)\exp(-x) = \exp(0) = 1.$$

So

$$\exp(-x) = \frac{1}{\exp(x)}$$

and

$$\exp(x+y) = \exp(y)\frac{1}{\exp(-x)} = \exp(y)\exp(x).$$

□

A neat argument, no?

Proposition 10.2 exp *is the only solution to the ordinary differential equation (ODE)*

$$\begin{cases} f'(x) & = f(x) \\ f(0) & = 1. \end{cases}$$

Proof Since $\frac{d}{dx}\exp(x) = \exp(x)$ and $\exp(0) = 1$, the exponential function is *one* solution of the ODE. Let $f(x)$ be *any* solution. Define $g(x) = f(x)\exp(-x)$.Then

$$\begin{aligned} g'(x) &= f'(x)\exp(-x) + f(x)\frac{d}{dx}[\exp(-x)] \\ &= f(x)\exp(-x) - f(x)\exp(-x) \\ &= 0. \end{aligned}$$

Hence for all x, $g(x) = g(0) = f(0)\exp(0) = 1 \cdot 1 = 1$. Thus

$$f(x)\exp(-x) = 1$$

and any solution $f(x)$ must be equal to $\exp(x)$. $\square$

It is obvious from the power series that $\exp(x) > 0$ for all $x \geq 0$. Since $\exp(-x) = 1/\exp(x)$, it follows that $\exp(x) > 0$ fopr all $x \in \mathbf{R}$.

Exercise 10.1 1. Prove that the range of exp is all of $\mathbf{R}_{>0}$. Hint: If $\exp(x) > 1$ for some x then $\exp(x^n)$ can be made as large, or as small, as you wish, by suitable choice of $n \in \mathbf{Z}$.

2. Show that $\exp : \mathbf{R} \to \mathbf{R}_{>0}$ is a bijection.

We would like to say next:

Proposition 10.3 *For all* $x \in \mathbf{R}$,

$$\exp(x) = e^x$$

where $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

But what does " e^x " mean? We all know that " e^2 " means "$e \times e$", but what about " e^π "? Operations of this kind are not part of nature - we need to *define* them.

It is easy to extend the simplest definition, that of raising a number to an *integer* power, to define what it means to raise a number to a rational power: first, for each $n \in \mathbf{N} \setminus \{0\}$ we define an "n'th root" function

$$\sqrt[n]{\ } : \mathbf{R}_{\geq 0} \to \mathbf{R}$$

as the inverse function of the (strictly increasing, infinitely differentiable) function

$$x \mapsto x^n.$$

Then for any $m/n \in \mathbf{Q}$, we set

$$x^{m/n} = \left(\sqrt[n]{x}\right)^m,$$

where we assume, as we may, that $n > 0$. But this does not tell us what e^π is. We could try approximation: choose a sequence of rational numbers x_n tending to π, and define

$$e^\pi = \lim_{n\to\infty} e^{x_n}.$$

We would have to show that the result is independent of the choice of sequence of x_n (i.e. depends only on its limit). This can be done. But then proving that the function $f(x) = a^x$ is differentiable, and finding its derivative, are rather hard. There is a much more elegant approach. First, we define $\ln : \mathbf{R}_{>0} \to \mathbf{R}$ as the inverse to the function $\exp : \mathbf{R} \to \mathbf{R}_{>0}$, which we know to be injective since its derivative is everywhere strictly positive, and surjective by Exercise 10.1. Then we make the following definition:

Definition 10.2 *For any $x \in \mathbf{R}$ and $a \in \mathbf{R}_{\geq 0}$,*

$$a^x := \exp(x \ln a).$$

Before anything else we should check that this agrees with the original definition of a^x where it applies, i.e. where $x \in \mathbf{Q}$. This is easy: because ln is the inverse of exp, and (by 10.1) exp turns addition into multiplication, it follows that ln turns multiplication into addition:

$$\ln(a \times b) = \ln\, a + \ln\, b,$$

from which we easily deduce that $\ln(a^m) = m \ln a$ (for $m \in \mathbf{N}$) and then that $\ln(a^{m/n}) = m/n \ln\, a$ for $m, n \in \mathbf{Z}$. Thus

$$\exp(\frac{m}{n} \ln x) = \exp(\ln(a^{m/n}) = a^{m/n},$$

the last equality because exp and ln are mutually inverse.

We have given a meaning to " e^x ", and we have shown that when $x \in \mathbf{Q}$ this new meaning coincides with the old meaning. Now that Proposition 10.3 is meaningful, we will prove it.

Proof In the light of Definition 10.2, Proposition 10.3 reduces to

$$\exp(x) = \exp(x \ln e). \tag{10.1}$$

But $\ln\, e = 1$, since $\exp(1) = e$; thus (10.1) is obvious! $\square$

Exercise 10.2 Show that Definition 10.2, and the definition of " a^x " sketched in the paragraph preceding Definition 10.2 agree with one another.

Proposition 10.4 *The natural logarithm function* $\ln : (0, \infty) \to \mathbf{R}$ *is differentiable and*

$$\ln'(x) = \frac{1}{x}.$$

Proof Since the differentiable bijective map $\exp(x)$ has $\exp'(x) \neq 0$ for all x, the differentiability of its inverse follows from the Inverse Function Theorem. And

$$\ln'(x) = \frac{1}{\exp'(y)}\Big|_{y=\ln(x)} = \frac{1}{\exp(\ln(x))} = \frac{1}{x}.$$

$$\begin{tikzcd} \ln x \quad \bullet \arrow[rr, bend left, "e^x"] & & \bullet \quad x \arrow[ll, bend left, "\ln x"] \end{tikzcd}$$

$\square$

Remark: In some computer programs, eg. gnuplot, $x^{\frac{1}{n}}$ is defined as following, $x^{\frac{1}{n}} = \exp(\ln(x^{\frac{1}{n}})) = \exp(\frac{1}{n} \ln x)$. Note that $\ln(x)$ is defined only for $x > 0$. This is the reason that typing in $(-2)^{\frac{1}{3}}$ returns an error message: it is not defined!

10.2 The Sine and Cosine Functions

We have defined

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n} \end{aligned}$$

Term by term differentiation shows that $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$.

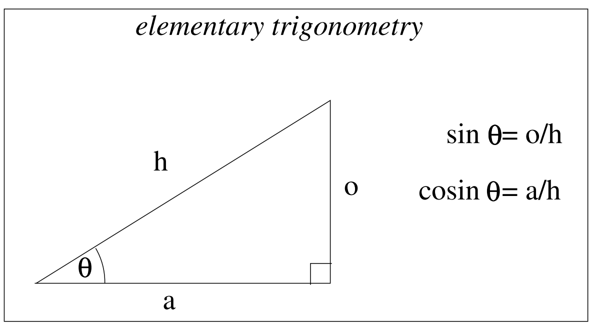
Exercise 10.3 For a fixed $y \in \mathbf{R}$, put

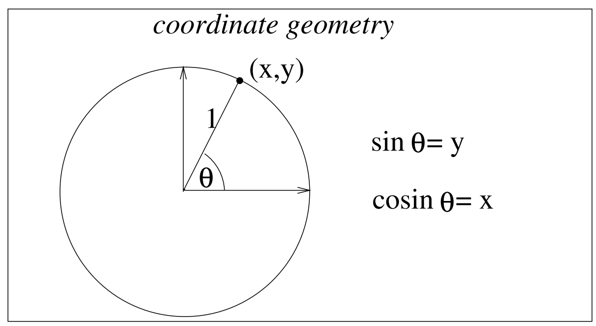
$$f(x) = \sin(x+y) - \sin x \cos y - \cos x \sin y.$$

Compute $f'(x)$ and $f''(x)$. Then let $E(x) = (f(x))^2 + (f'(x))^2$. What is E'? Apply the Mean Value Theorem to E, and hence prove the addition formulae

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y. \end{aligned}$$

The sine and cosine functions defined by means of power series behave very well, and it is gratifying to be able to prove things about them so easily. But what relation do they bear to the sine and cosine functions defined in elementary geometry?





We cannot answer this question now, but will be able to answer it after discussing Taylor's Theorem in the next chapter.

Chapter 11

Taylor's Theorem and Taylor Series

We state a few motivating problems for Taylor's Theorem. A polynomial of degree at most n is a fuction of the form:

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

1. The problem of "polynomial interpolation " is to find a polynomial of degree n taking prescribed values c_i at a certain number of points x_i, for $i = 1, \ldots, k$. Such a polynomial exists provided $k \leq n+1$. If $k = n+1$, there is only one solution, since given any two solutions P_1 and P_2, their difference $P_1 - P_2$ is a polynomial of degree $\leq n$ which vanishes at $n+1$ points, and thus is identically zero. If $k > n+1$ then there is usually no solution. If the c_i are the values taken by some given function f at the points x_i (e.g. $f(x_i)$ is the electical resistance of a material at temperature x_i) then P is a *degree n polynomial approximation to f.*

2. A slightly more sophisticated version of the problem involves prescribing not only the values of P at k points, but the values of some of its derivatives also.

3. *Our problem:* The problem that gives rise to Taylor's Theorem is that of finding a polynomial P of degree n such that

$$P(x_0) = c_0, P'(x_0) = c_1, P''(x_0) = c_2, \ldots, P^{(n)}(x_0) = c_n$$

for prescribed values $c_0, \ldots, c_n$. In fact this is easily solved, as we will shortly see, and has a unique solution. Taylor's Theorem itself is concerned with the following question: if we are given a function f, and we find a degree n polynomial P whose value and first n derivatives coincide with those of f at the point x_0, how well does P approximate f in the neighbourhood of x_0?

Lemma 11.1 *Suppose f is n times differentiable on (a, b). Let $P_n(x)$ be a degree n polynomial such that*

$$P_n(x_0) = f(x_0),\ P_n'(x_0) = f'(x_0), \dots, P_n^{(n)}(x_0) = f^{(n)}(x_0).$$

Then

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \end{aligned}$$

Proof Every polynomial of degree n can be written in the following form:

$$P_n(x) = \frac{1}{n!}a_n(x - x_0)^n + \frac{1}{(n-1)!}a_{n-1}(x - x_0)^{n-1} + \cdots + a_1(x - x_0) + a_0. \tag{11.1}$$

Set $x = x_0$ to obtain $a_0 = P_n(x_0)$. For $k = 1, \dots, n$, differentiating (11.1) k times shows that $P_n^{(k)}(x_0) = a_k$. If P_n is a degree n polynomial satisfying the conditions of the Lemma, then we must have $a_k = f^{(k)}(x_0)$, as required. □

The polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}$$

is called the *degree n Taylor polynomial of f about x_0*. **Questions:**

1. Let $R_n(x) = f(x) - P_n(x)$ be the "error term". It measures how well the polynomial P_n approximates the value of f. How large is the error term? Taylor's Theorem is concerned with estimating the value of the error $R_n(x)$.

2. Is it true that $\lim_{n\to\infty} R_n(x) = 0$? If so, then

$$f(x) = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

and we have a "power series expansion" for f.

3. Does every infinitely differentiable function have a power series expansion?

Definition 11.1 *If f is C^∞ on its domain, the infinite series*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$$

is called the Taylor series for f about x_0, or around x_0.

Does the Taylor series of f about x_0 converge on some neighbourhood of x_0? If so, it defines an infinitely differentiable function on this neighbourhood. Is this function equal to f?

The answer to both questions are yes for some functions and no for some others.

Definition 11.2 *If f is C^∞ on its domain (a,b) and $x_0 \in (a,b)$, we say Taylor's formula holds for f near x_0 if for all x in some neighbourhood of x_0,*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}.$$

The following is an example of a C^∞ function whose Taylor series converges, but does not converge to f:

Example 11.1 (Cauchy's Example (1823))

$$f(x) = \begin{cases} \exp(-\frac{1}{x^2}), & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

It is easy to see that if $x \neq 0$ then $f(x) \neq 0$. Below we sketch a proof that $f^{(k)}(0) = 0$ for all k. The Taylor series for f about 0 is therefore:

$$\sum_{k=0}^{\infty} \frac{0}{k!} x^k = 0 + 0 + 0 + \dots.$$

This Taylor series converges everywhere, obviously, and the function it defines is the zero function. So it does not converge to $f(x)$ unless $x = 0$.

How do we show that $f^{(k)}(0) = 0$ for all k? Let $y = \frac{1}{x}$,

$$f'_+(0) = \lim_{x \to 0+} \frac{\exp(-\frac{1}{x^2})}{x} = \lim_{y \to +\infty} \frac{y}{\exp(y^2)}.$$

Since $\exp(y^2) \geq y^2$, $\lim_{y\to+\infty} \frac{y}{\exp(y^2)} = 0$ and $\lim_{y\to-\infty} \frac{y}{\exp(y^2)} = 0$. It follows that

$$f'_+(0) = f'_(0) = 0.$$

A similar argument gives the conclusion for $k > 1$. An induction is needed.

Exercise 11.2 1. If $Q = a_0 + a_1 + \cdots + a_m x^m$ is a polynomial of degree m, $m = 0, 1, 2, \ldots$, show that

$$\lim_{y \to +\infty} \frac{Q(y)}{\exp(y^2)} = 0.$$

2. Show that

$$\lim_{x \to 0} \frac{Q(\frac{1}{x}) \exp(-\frac{1}{x^2})}{x} = 0.$$

3. Compute the derivative of $f^{(k)}(x)$ for $x \neq 0$. For example,

$$\begin{aligned}
f'(x) &= -2x^{-3} \exp(-\frac{1}{x^2}) \\
f''(x) &= [-2x^{-3}]^2 \exp(-\frac{1}{x^2}) + 3!x^{-4} \exp(-\frac{1}{x^2}) \\
f^{'''}(x) &= [-2x^{-3}]^3 \exp(-\frac{1}{x^2}) - 4!x^{-5} \exp(-\frac{1}{x^2}) - 2(3!)x^{-6} \exp(-\frac{1}{x^2}).
\end{aligned}$$

By induction show that

$$f^{(k)}(x) = Q_k(\frac{1}{x}) \exp(-\frac{1}{x^2})$$

Where $Q_k(y)$ is a polynomial of degree $3k$ with leading term $(-1)^k 2^k y^{3k}$ and

$$Q_k(y) = (-1)^k 2^k y^{3k} + \ldots.$$

Note that

$$f^{(k+1)}(0) = \lim_{x \to 0} \frac{f^{(k)}(x)}{x} = \lim_{x \to 0} \frac{Q_k(\frac{1}{x}) \exp(-\frac{1}{x^2})}{x}.$$

4. Show that $f^{(k+1)}(0) = 0$.

Remark 11.1 By Taylor's Theorem below, if

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi_n) \to 0$$

where ξ_n is between 0 and x, Taylor's formula holds. It follows that, for the function above, $\lim_{n\to\infty} R_n(x)$ either does not exist or if it exists the limit cannot be 0. Convince yourself of this (without rigorous proof) by observing that $Q_{n+1}(y)$ contains a term of the form $(-1)^{n+1}(n+2)!y^{-(n+2)}$. And indeed

$$\frac{x^{n+1}}{(n+1)!}(-1)^{n+1}(n+2)!\xi_n^{-n-2}$$

may not converge to 0 as $n \to \infty$.

Definition 11.3 ** A C^∞ function $f : (a,b) \to \mathbf{R}$ is said to be* real analytic *if for each $x_0 \in (a,b)$, its Taylor series about x_0 converges, and converges to f, in some neighbourhood of x_0.*

The preceding example shows that the function e^{-1/x^2} is not real analytic in any interval containing 0, even though it is infinitely differentiable. Complex analytic functions are studied in the third year Complex Analysis course. Surprisingly, *every* complex differentiable function is complex analytic.

Example 11.3 Find the Taylor series for $\exp(x) = \sum_{k=0}^\infty \frac{x^k}{k!}$ about $x_0 = 1$. For which x does the Taylor series converge? For which x does Taylor's formula hold?

Solution. For all k, $\exp^{(k)}(1) = e^1 = e$. So the Taylor series for exp about 1 is

$$\sum_{k=0}^\infty \frac{\exp^{(k)}(1)(x-1)^k}{k!} = \sum_{k=0}^\infty \frac{e(x-1)^k}{k!} = e\sum_{k=0}^\infty \frac{(x-1)^k}{k!}.$$

The radius of convergence of this series is $+\infty$. Hence the Taylor series converges everywhere.

Furthermore, since $\exp(x) = \sum_{k=0}^\infty \frac{x^k}{k!}$, it follows that

$$\sum_{k=0}^\infty \frac{(x-1)^k}{k!} = \exp(x-1)$$

and thus

$$e\sum_{k=0}^{\infty}\frac{(x-1)^k}{k!} = e\exp(x-1) = \exp(1)\exp(x-1) = \exp(x)$$

and Taylor's formula holds.

Exercise 11.4 Show that exp is real analytic on all of $\mathbf{R}$.

11.1 More on Power Series

Definition 11.4 *If $x_0 \in \mathbf{R}$, a formal power series centred at x_0 is an expression of the form:*

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots.$$

Example 11.5 Take $\sum_{n=0}^{\infty}\frac{1}{3^n}(x-2)^n$. If $x=1$,

$$\sum_{n=0}^{\infty}\frac{1}{3^n}(x-2)^n = \sum_{n=0}^{\infty}\frac{1}{3^n}(1-2)^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^n}$$

is a convergent series.

What we learnt about power series $\sum_{n=0}^{\infty} a_n x^n$ can be applied here. For example,

1. There is a radius of convergence R such that the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges when $|x-x_0| < R$, and diverges when $|x-x_0| > R$.

2. R can be determined by the Hadamard Theorem:

$$R = \frac{1}{r}, \qquad r = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

3. The open interval of convergence for $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is

$$(x_0 - R, x_0 + R).$$

The functions

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty} a_n x^n$$

are related by

$$f(x) = g(x-x_0).$$

4. The term by term differentiation theorem holds for x in (x_0-R, x_0+R) and

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1} = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots.$$

Example 11.6 Consider $\sum_{n=0}^{\infty} \frac{1}{3^n}(x-2)^n$.

Then

$$\limsup (\frac{1}{3^n})^{\frac{1}{n}} = \frac{1}{3}, \qquad \text{so} \qquad R = 3.$$

The power series converges for all x with $|x-2| < 3$.
For $x \in (-1, 5)$, $f(x) = \sum_{n=0}^{\infty} \frac{1}{3^n}(x-2)^n$ is differentiable and

$$f'(x) = \sum_{n=0}^{\infty} n \frac{1}{3^n}(x-2)^{n-1}.$$

11.2 Uniqueness of Power Series Expansion

Example 11.7 What is the Taylor's series for $\sin(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$ about $x = 0$?

Solution. $\sin(0) = 0$, $\sin'(0) = \cos 0 = 1$, $\sin^{(2)}(0) = -\sin 0 = 0$, $\cos^{(3)}(0) = -\cos 0 = -1$, By induction, $\sin^{(2n)}(0) = 0$, $\sin^{(2n+1)} = (-1)^n$. Hence the Taylor series of $\sin(x)$ at 0 is

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)!} x^{2k+1}.$$

This series converges everywhere and equals $\sin(x)$.

The following proposition shows that we would have needed no computation to compute the Taylor series of $\sin x$ at 0:

Proposition 11.2 (Uniqueness of Power Series expansion) *Let $x_0 \in (a, b)$. Suppose that $f : (a, b) \subset (x_0 - R, x_0 + R) \to \mathbf{R}$ is such that*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where R is the radius of convergence. Then

$$a_0 = f(x_0),\ a_1 = f'(x_0), \dots, a_k = \frac{f^{(k)}(x_0)}{k!}, \dots.$$

Proof Take $x = x_0$ in

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots,$$

it is clear that $f(x_0) = a_0$. By term by term differentiation,

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \dots.$$

Take $x = x_0$ to see that $f'(x_0) = a_1$. Iterating this argument, we get

$$f^{(k)}(x) = a_k k! + a_{k+1}(k+1)!(x - x_0) + a_{k+2}\frac{(k+2)!}{2!}(x - x_0)^2 + a_{k+3}\frac{(k+3)!}{3!}(x - x_0)^3 + \dots.$$

Take $x = x_0$ to see that

$$a_k = \frac{f^{(k)}(x_0)}{k!}.$$

□

Moral of the proposition: if the function is already in the form of a power series centred at x_0, this is the Taylor series of f about x_0.

Example 11.8 What is the Taylor series of $f(x) = x^2 + x$ about $x_0 = 2$?

Solution 1. $f(2) = 6$, $f'(2) = 5$, $f''(2) = 2$. Since $f'''(x) = 0$ for all x, the Taylor series about $x_0 = 2$ is

$$6 + \frac{5}{1}(x - 2) + \frac{2}{2!}(x - 2) = 6 + 5(x - 2) + (x - 2)^2.$$

Solution 2. On the other hand, we may re-write

$$f(x) = x^2 + x = (x - 2 + 2)^2 + (x - 2 + 2) = (x - 2)^2 + 5(x - 2) + 6.$$

The Taylor series for f at $x_0 = 2$ is $(x - 2)^2 + 5(x - 2) + 6$.

The Taylor series is identical to f and so Taylor's formula holds.

Example 11.9 Find the Taylor series for $f(x) = \ln(x)$ about $x_0 = 1$.

Solution. If $x > 0$,

$$
\begin{aligned}
f(1) = 0 & \\
f'(x) &= \frac{1}{x}, \qquad f'(1) = 1 \\
f''(x) &= -\frac{1}{x^2}, \qquad f''(1) = -1 \\
f^{(3)}(x) &= 2\frac{1}{x^3}, \qquad f'''(1) = -2 \\
&\dots \\
f^{(k)}(x) &= (-1)^{k+1}(k-1)!\frac{1}{x^k}, \qquad f^{(k)}(1) = (-1)^{(k+1)}(k-1)! \\
f^{(k+1)}(x) &= (-1)^{k+2}k!\frac{1}{x^{k+1}}.
\end{aligned}
$$

Hence the Taylor series is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{k}(x-1)^k = (x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\dots.$$

It has radius of convergence $R = 1$.

Question: Does Taylor's formula hold? i.e. is it true that

$$\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{k}(x-1)^k?$$

If we write $x - 1 = y$, we are asking whether

$$\ln(1+y) = \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{k} y^k.$$

We will need Taylor's Theorem (which tells us how good is the approximation) to answer this question.

Remark: We could borrow a little from Analysis III, Since $\sum_{k=0}^{\infty}(-x)^k = \frac{1}{1+x}$, term by term integration (which we are not yet able to justify) would give:

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = \ln(1+x).$$

11.3 Taylor's Theorem in Integral Form*

This section is not included in the lectures nor in this module. The remainder term $R_n(x)$ in the integral form is the best formulation. But we must know Theory of Integration (Analysis III). With what we learnt in school and a little bit flexibility we should be able to digest the information. Please read on.

Theorem 11.3 (Taylor's Theorem with remainder in integral form*) *If f is C^{n+1} on $[a,x]$, then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k}(x-a)^k + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

Thus

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

Proof By the fundamental Theorem of calculus from Analysis III:

$$f(x) = f(a) + \int_a^x f'(t)dt. \tag{11.2}$$

Integration by parts gives

$$\begin{aligned}
\int_a^x f'(t)dt &= -\int_a^x f'(t)\frac{d}{dt}(x-t)dt \\
&= \big[-f'(t)(x-t)\big]_a^x + \int_a^x (x-t)f''(t)dt \\
&= f'(a)(x-a) + \int_a^x (x-t)f''(t)dt.
\end{aligned}$$

So by (11.2),

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)dt.$$

This proves the theorem when $n=1$. Induction on n proves the case for all n:

suppose that

$$f(x) = P_n(x) + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

Then, using the same trick as in the case where $n = 1$, we have

$$\begin{aligned} f(x) &= P_n(x) + \left[-f^{(n+1)}(t) \frac{(x-t)^{n+1}}{(n+1)!} \right]_a^x + \int_a^x f^{(n+2)}(t) \frac{(x-t)^{n+1}}{(n+1)!} dt \\ &= P_{n+1}(x) + \int_a^x f^{(n+2)}(t) \frac{(x-t)^{n+1}}{(n+1)!} dt \\ &= P_{n+1}(x) + R_{n+1}(x). \end{aligned}$$

□

Lemma 11.4 (Intermediate Value Theorem in Integral Form*) *If g is continuous on $[a, b]$ and f is positive everywhere and integrable, then for some $c \in (a, b)$,*

$$\int_a^b f(x)g(x)dx = g(c) \int_a^b f(x)dx.$$

Proof By the extreme value theorem, there are numbers $x_1, x_2 \in [a, b]$ such that $m = g(x_1), M = g(x_2)$ and

$$m \le g(x) \le M.$$

Since $f(x) \ge 0$, multiplication by $f(x)$ preserves the order:

$$mf(x) \le f(x)g(x) \le Mf(x).$$

Integrate each term in the above inequality from a to b:

$$m \int_a^b f(x)dx \le \int_a^b f(x)g(x)dx \le M \int_a^b f(x)dx$$

If $\int_a^b f(x)dx = 0$, the theorem is proved. Otherwise $\int_a^b f(x)dx >= 0$, and

$$m \le \frac{1}{\int_a^b f(x)dx} \int_a^b f(x)g(x)dx \le M.$$

By the Intermediate Value Theorem , applied to g, there exists $c \in [x_1, x_2]$ such that

$$g(c) = \frac{1}{\int_a^b f(x)dx} \int_a^b f(x)g(x)dx.$$

□

Remark 11.2 It is necessary for the lemma that f does not change sign. For example if $f(x) = \sin(x)$, $x \in [0, 2\pi]$, $\int_0^{2\pi} \sin(x)dx = 0$, but $\int_0^{2\pi} (\sin x)(\sin x)dx = \frac{1}{2}\int_0^{2\pi}(1 - \cos(2x))dx \neq 0$.

By this lemma, the integral remainder term in Theorem 11.3 can take the folllowing form:

$$\begin{aligned} R_n(x) &= \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \\ &= f^{(n+1)}(c) \int_a^x \frac{(x-t)^n}{n!} dt \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}. \end{aligned}$$

This is the **Lagrange form** of the remainder term $R_n(x)$ in Taylor's Theorem, as shown in 11.5 below. We have just deduced Theorem 11.5 from Theorem 11.3. In the next section which we will prove it again by the Mean Value Theorem, without borrowing theorems from the theory of integration.

The **Cauchy form** for the remainder $R_n(x)$ is

$$R_n = f^{(n+1)}(c)\frac{(x-c)^n}{n!}(x-a).$$

This can be obtained in the same way:

$$\begin{aligned} R_n(x) &= \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \\ &= f^{(n+1)}(c)(x-c)^n \int_a^x \frac{1}{n!} dt \\ &= f^{(n+1)}(c)(x-c)^n \frac{x-a}{n!}. \end{aligned}$$

Other integral techniques:

Example 11.10 For $|x| < 1$, by term by term integration (proved in second year modules)

$$\begin{aligned}
\arctan x &= \int_0^x \frac{dy}{1+y^2} = \int_0^x \sum_{n=0}^{\infty}(-y^2)^n dy \\
&= \sum_{n=0}^{\infty}\int_0^x (-y^2)^n dy \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{y^{2n+1}}{2n+1}\Big|_{y=0}^{y=x} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}
\end{aligned}$$

11.4 Taylor's Theorem with remainder in Lagrange form

If f is continuous on $[a,b]$ and differentiable on (a,b), the Mean Value Theorem states that

$$f(b) = f(a) + f'(c)(b-a)$$

for some $c \in (a,b)$. We could treat $f(a) = P_0$, a polynomial of degree 0.

We have seen earlier in Lemma 9.16, iterated use of the Mean Value Theorem gives nice estimates. Let us try that.

Let $f : [a,b] \to \mathbf{R}$ be continuously differentiable and twice differentiable on (a,b). Consider

$$f(b) = f(x) + f'(x)(b-x) + error.$$

The quantity $f(x) + f'(x)(b-x)$ describes the value at b of the tangent line of f at x. We now vary x. We guess that the error is of the order $(b-x)^2$. Define

$$g(x) := f(b) - f(x) - f'(x)(b-x) - A(b-x)^2$$

with A a number to be determined so that we could Apply Rolle's Theorem.

Note that $g(b) = 0$ and $g(a) = f(b) - f(a) - f'(a)(b-a) + A(b-a)^2$. Set

$$A = \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}.$$

Apply Rolle's theorem to g we see that there is $\xi \in (a,b)$ such that $g'(\xi) = 0$. Since,

$$\begin{aligned} g'(x) &= -f'(x) - [f''(x)(b-x) - f'(x)] + 2\frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}(b-x) \\ &= -f''(x)(b-x) + 2\frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}(b-x). \end{aligned}$$

we see that

$$\frac{1}{2}f''(\xi) = \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}.$$

This gives

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2}f''(\xi)(b-a)^2.$$

The following is a theorem of Lagrange (1797). To prove it we apply MVT to $f^{(n)}$ and hence we need to assume that $f^{(n)}$ satisfies the conditions of the Mean Value Theorem. By f is n times continuously differentiable on $[x_0, x]$ we mean that its nth derivative is continuous otherwise denoted by $f \in C^n([a,b])$.

Theorem 11.5 (Taylor's theorem with Lagrange Remainder Form) *1. Let $x > x_0$. Suppose that f is n times continuously differentiable on $[x_0, x]$ and $n+1$ times differentiable on (x_0, x). Then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x)$$

where

$$R_n(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{n+1}(\xi)$$

some $\xi \in (x_0, x)$.

2. The conclusion holds also for $x < x_0$, if f is $(n+1)$ times continuously differentiable on $[x, x_0]$ and $n+1$ times differentiable on (x, x_0).

Proof Let us vary the starting point of the interval and consider $[y, x]$ for any $y \in [x_0, x]$. We will regard x as fixed (it does not move!).

We are interested in the function with variable y:

$$g(y) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(y)(x-y)^k}{k!}.$$

In long form,

$$g(y) = f(x) - [f(y) + f'(y)(x-y) + f''(y)(x-y)^2/2 + \cdots + \frac{f^{(n)}(y)(x-y)^n}{n!}].$$

Then $g(x_0) = R_n(x)$ and $g(x) = 0$. Define

$$h(y) = g(y) - A(x-y)^{n+1}$$

where

$$A = \frac{g(x_0)}{(x-x_0)^{n+1}}.$$

We check that h satisfies the conditions of Rolle's Theorem on $[x_0, x]$:

- $h(x_0) = g(x_0) - A(x-x_0)^{n+1} = 0$
- $h(x) = g(x) = 0$.
- h is continuous on $[x_0, x]$ and differentiable on (x_0, x).

Hence, by Rolle's Theorem, there exists some $\xi \in (x_0, x)$ such that $h'(\xi) = 0$.

Now we calculate $h'(y)$. First, by the product rule,

$$\begin{aligned}
g'(y) &= -\frac{d}{dy}\sum_{k=0}^{n}\frac{f^{(k)}(y)(x-y)^k}{k!} \\
&= -\sum_{k=0}^{n}\frac{f^{(k+1)}(y)(x-y)^k}{k!} + \sum_{k=1}^{n}\frac{f^{(k)}(y)k(x-y)^{k-1}}{k!} \\
&= -\sum_{k=0}^{n}\frac{f^{(k+1)}(y)(x-y)^k}{k!} + \sum_{k=1}^{n}\frac{f^{(k)}(y)(x-y)^{k-1}}{(k-1)!} \\
&= -\frac{f^{(n+1)}(y)(x-y)^n}{n!}.
\end{aligned}$$

Next,

$$\frac{d}{dy}\left(A(x-y)^{n+1}\right) = (n+1)A(x-y)^n.$$

Thus,

$$h'(y) = -\frac{f^{(n+1)}(y)(x-y)^n}{n!} - (n+1)A(x-y)^n$$

and the vanishing of $h'(\xi)$ means that

$$\frac{f^{(n+1)}(\xi)(x-\xi)^n}{n!} = \left(\frac{g(x_0)}{(x-x_0)^{n+1}}\right)(n+1)(x-\xi)^n$$

(we have replaced A by $g(x_0)/(x-x_0)^{n+1}$). Since $R_n(x) = g(x_0)$, we deduce that

$$R_n(x) = \frac{(x-x_0)^{(n+1)}}{(n+1)!} f^{(n+1)}(\xi).$$

$\square$

We can also give the following version of Taylor's theorem.

Theorem 11.6 (Taylor's theorem, double sided version) *Let $x_0 \in (a,b)$. Suppose that f is n times continuously differentiable on $[a,b]$ and $n+1$ times differentiable on (a,b). Then for $x \in [a,b]$, there exists ξ between x_0 and x such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!} + R_n(x)$$

where

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{n+1}(\xi)$$

The only difference between this and Theorem 11.5 is that it allows x to be greater than x_0 or less than x_0. Proof for the case $x < x_0$ is very similar to the proof for the case $x > x_0$. It would be a good idea for you to write it out!

Problem 11.7 Let $f \in C^{(n+1)}(a,b)$, hence the assumptions of Taylor's theorem hold. Letting

$$P_n^f(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!},$$

be the Taylor Polynomial of degree n. If $f^{(k)}(x_0) = 0$ for $k = 0, 1, \ldots, n$, then $P_n^f(x) \equiv 0$, hence the n-th order polynomial approximation for f is the zero function ! The point $\xi \in (a,b)$ may change with x and hence should be treated as a function of x so

$$f(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{n+1}(\xi(x)).$$

Question: does the function $f^{n+1}(\xi(x))$ depend continuously on x? Note that

$$f^{n+1}(\xi(x)) = (n+1)! \frac{f(x)}{(x-x_0)^{(n+1)}}.$$

Given examples of functions for which the above function is not continuous at x_0! Find a set of assumptions under which $f^{(n+1)}(\xi(x))$ is continuous at x_0. You may need to read the rest of the lecture notes before answering this question.

Corollary 11.8 *Let $x_0 \in (a,b)$. If f is C^∞ on $[a,b]$ and the derivatives of f are uniformly bounded on (a,b), i.e. there exists some K such that*

$$\text{for all } k \text{ and for all } x \in [a,b], \quad |f^{(n)}(x)| \le K$$

then Taylor's formula holds, i.e. for each $x, x_0 \in [a,b]$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}, \qquad x \in [a,b].$$

In other words f has a Taylor series expansion at every point of (a,b). In yet other words, f is real analytic in $[a,b]$.

Proof For each $x \in [a,b]$,

$$\begin{aligned} |R_n(x)| &= \left| \frac{(x-x_0)^{n+1}}{(n+1)!} f^{n+1}(\xi) \right| \\ &\le K \frac{|x-x_0|^{n+1}}{(n+1)!} \end{aligned}$$

and this tends to 0 as $n \to \infty$. This means

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!} = f(x)$$

as required. □

Now we can answer the question we posed at the end of Section 10. Consider the sine and cosine functions defined using elementary geometry. We have seen in Example 6.3 that cosine is the derivative of sine and the derivative of cosine is minus sine. Since both |sine| and |cosine| are bounded by 1 on all of $\mathbf{R}$, both sine and cosine satisfy the conditions of Corollary 11.8, with $K = 1$. It follows that both sine and cosine are analytic on all of $\mathbf{R}$. Power series 2 and 3 of Definition 10.1 are the Taylor series of sine and cosine about $x_0 = 0$. Corollary 11.8 tells us that they converge to sine and cosine. So the definition by means of power series gives the same function as the definition in elementary geometry. Just in case you think that elementary geometry is for children and that grown-ups prefer power series, try proving, directly from the power series definition, that sine and cosine are periodic with period 2π.

Example 11.11 Compute $\sin(1)$ to the precision 0.001.

Solution For some $\xi \in (0,1)$,

$$|R_n(1)| = \left|\frac{\sin^{(n+1)}(\xi)}{(n+1)!}1^{n+1}\right| \le \frac{1}{(n+1)!}.$$

We wish to have

$$\frac{1}{(n+1)!} \le 0.001 = \frac{1}{1000}.$$

Take n such that $n! \ge 1000$. Take $n = 7$, $7! = 5040$ is sufficient. Then

$$\sin(1) \sim [x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}]|_{x=1} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}.$$

Now use your calculator if needed.

Example 11.12 Show that $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$.

By Taylor's Theorem $\cos x - 1 = \frac{x^2}{2} + R_3(x)$. For $x \to 0$ we may assume that $x \in [-1,1]$. Then for some $c \in (-1,1)$,

$$|R_3(x)| = |\frac{\cos^{(3)}(c)}{3!}x^3| = |\frac{\sin(c)}{3!}x^3| \le |x^3|.$$

It follows that

$$\left|\frac{\cos x - 1}{x}\right| \le \frac{x^2/2 + |x|^3}{|x|} = |x/2| + |x^2| \to 0,$$

as $x \to 0$.

How do we prove that $\lim_{n\to\infty} \frac{r^n}{n!} = 0$ for any $r > 0$? Note that

$$\sum_{n=0}^{\infty} \frac{r^n}{n!}$$

convergenes by the Ratio Test, and so its individual terms $\frac{r^n}{n!}$ must converge to 0.

Example 11.13 Give the Taylor series for e^x about $x_0 = 0$ using only $(e^x)' = e^x$ and $e^0 = 1$. Since $\frac{d^n}{dx^n}e^x = e^x$ taking value 0 at $x = 0$, the Taylor series is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The Taylor series has radius of convergence ∞. For any $x \in \mathbf{R}$, there is $\xi \in (-x, x)$ s.t.

$$|R_n(x)| = \left|\frac{x^{n+1}}{(n+1)!}e^{\xi}\right| \leq \max(e^x, e^{-x})|\frac{x^{n+1}}{(n+1)!}| \to 0$$

as $n \to \infty$. So

$$e^x = \sum_{k=0\infty} \frac{x^k}{k!} = \exp(x)$$

for all x.

Example 11.14 The Taylor series for $\ln(1+x)$ about 0 is

$$\sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{k} x^k$$

with $R = 1$. Do we have, for $x \in (-1, 1)$,

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{k} x^k?$$

Solution. Let $f(x) = \ln(1+x)$, then

$$f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{(1+x)^n}.$$

and

$$\begin{aligned} |R_n(x)| &= \left|\frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}\right| \\ &= \left|\frac{(-1)^{n+2} n!}{(n+1)!(1+\xi)^{n+1}} x^{n+1}\right| \\ &= \frac{1}{(n+1)} \left|\frac{x}{1+\xi}\right|^{n+1} \end{aligned}$$

If $0 < x < 1$, then $\xi \in (0, x)$ and $\left|\frac{x}{1+\xi}\right| < 1$.

$$\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{1}{(n+1)} = 0.$$

*For the case of $-1 < x < 0$, we use the integral form of the remainder term $R_n(x)$. Since

$$f^{(n+1)}(x) = (-1)^n n! \frac{1}{(1+x)^{n+1}},$$

and $x < 0$,

$$\begin{aligned} R_n(x) &= \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \\ &= (-1)^n \int_0^x (x-t)^n \frac{1}{(1+t)^{n+1}} dt \\ &= -\int_x^0 (t-x)^n \frac{1}{(1+t)^{n+1}} dt. \end{aligned}$$

Since

$$\frac{d}{dt}\left(\frac{t-x}{1+t}\right) = \frac{(1+t)-(t-x)}{(1+t)^2} = \frac{(1+x)}{(1+t)^2} > 0,$$

$\frac{t-x}{1+t}$ is an increasing function in t and

$$\max_{x \le t \le 0} \frac{t-x}{1+t} = \frac{0-x}{1+0} = -x > 0.$$

$$\min_{x \le t \le 0} \frac{t-x}{1+t} = \frac{x-x}{1+x} = 0.$$

Note that $-x > 0$.

$$\begin{aligned} |R_n(x)| &= \int_x^0 (x-t)^n \frac{1}{(1+t)^{n+1}} dt \\ &\le \int_x^0 \frac{(-x)^n}{(1+t)} dt \\ &= (-x)^n[-\ln(1+x)]. \end{aligned}$$

Since $0 < x+1 < 1$, $\ln(1+x) < 0$ and $0 < -x < 1$. It follows that

$$\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} (-x)^n[-\ln(1+x)] = 0.$$

Thus Taylor's formula hold for $x \in (-1, 1)$.

11.4.1 Cauchy's Mean Value Theorem

Lemma 11.9 (Cauchy's Mean Value Theorem) *If f and g are continuous on $[a,b]$ and differentiable on (a,b) then there exists $c \in (a,b)$ with*

$$(f(b)-f(a))g'(c) = f'(c)[g(b)-g(a)].$$

If $g'(c) \neq 0 \neq g(b)-g(a)$, then

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof Set

$$h(x) = g(x)[f(b)-f(a)] - f(x)[g(b)-g(a)].$$

Then $h(a) = h(b)$, h is continuous on $[a,b]$ and differentiable on (a,b). By Rolle's Theorem, there is a point $c \in (a,b)$ such that

$$0 = h'(c) = g'(c)[f(b)-f(a)] - f'(c)[g(b)-g(a)],$$

hence the conclusion. □

Remark 11.3 Analytic intuition: given functions f and g, find a function h of the form

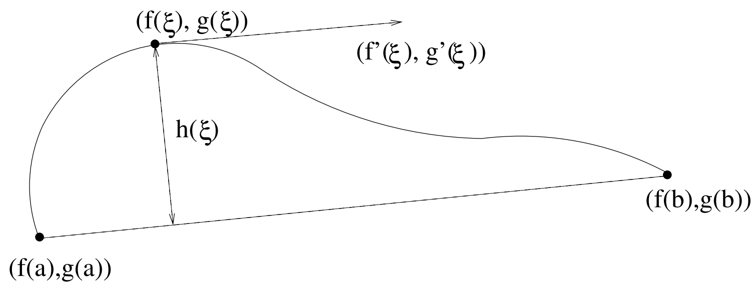
$$h(x) = \alpha f(x) + \beta g(x)$$

such that $h(a) = h(b)$. Then

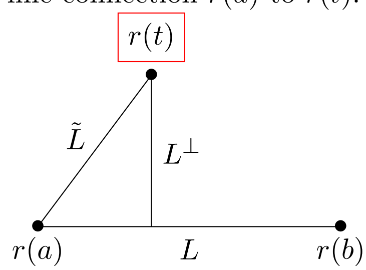
$$\frac{\beta}{\alpha} = \frac{f(a)-f(b)}{g(b)-f(a)}.$$

Geometric Interpretation: Consider $t \in [a,b] \mapsto (f(t), g(t))$, a curve in the plane. Let L be the line connecting the two points with coordinates $(f(a), g(a))$ and $(f(b), g(b))$. We look for a point c on the graph where its distance from the straight line L is a local extrema.

Proposition 11.10 *The function r, $t \in [a,b] \mapsto r(t) \equiv (f(t), g(t))$ parametrises a curve C in the plane. Let L be the line connecting the two points with coordinates $r(a) = (f(a), g(a))$ and $r(b) = (f(b), g(b))$ in the plane. Let $h(t)$ be the distance of a point $r(t)$ to the line L. Then if ξ is as in Cauchy's Mean Value Theorem, then ξ is a critical point of h. Equivalently, the tangent to the curve C at the point $r(\xi)$ is parallel to the line joining the points $r(a) = (f(a), g(a))$ and $r(b) = (f(b), g(b))$.*



Proof Let $L^{\perp}$ be the line from $r(t)$ that is perpendicular to L. Let $\tilde{L}$ be the line connection $r(a)$ to $r(t)$. Then $h(t)$ is the projection of $\tilde{L}$ to $L^{\perp}$.



The direction of L is in the direction of the vector $u := (f(b)-f(a), g(a), g(b))$. The direction n of $L^{\perp}$ is orthogonal to u:

$$n = (g(b) - g(a), -f(b) + f(a)).$$

Th projection of the vector $h(t) - r(a)$ to line $\tilde{L}$ is: $\frac{1}{|n|} n \cdot (r(t) - r(a))$, using the dot product, and

$$h(t) = \frac{1}{|n|} \left| n \cdot (r(t) - r(a)) \right|,$$

given the the dot product. Now $|n|$ is a constant of t and

$$\begin{aligned} n \cdot (r(t) - r(a)) &= [g(b) - g(a)]f(t) + [-f(b) + f(a)]g(t) \\ &\quad -[g(b) - g(a)]f(a) + [-f(b) + f(a)]g(a). \end{aligned}$$

We investigate critical point of $h(t)$. Set $h'(t) = 0$ which implies that

$$\frac{f'(t)}{g'(t)} = \frac{f(a) - f(b)}{g(a) - g(b)}.$$

□

Using Cauchy's Mean Value Theorem we now give a third proof of Taylor's Theorem (Theorem 11.5)

Proof We compare

$$g_n(x) := \frac{(x-x_0)^{n+1}}{(n+1)!}$$

with the remainder term

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

The first n derivatives of $g_n(x)$ vanish at x_0 by construction.

$$\begin{array}{rcl} g_n(x_0) & = & 0 \\ g_n'(x_0) & = & (n+1)\frac{(x-x_0)^n}{n!}\Big|_{x=x_0} = 0 \\ & \cdots & \\ g_n^{(n)}(x_0) & = & [(x-x_0)]_{x=x_0} = 0 \\ g_n^{(n+1)}(x_0) & = & 1. \end{array}$$

Let

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

then $P_n(x_0) = f(x_0)$,

$$P_n^{(k)}(x_0) = f^{(k)}(x_0)$$

for all $k \le n$ and $P_n^{(n+1)}(x) = 0$. This gives

$$\begin{array}{rcl} R_n(x_0) & = & 0 \\ R_n'(x_0) & = & 0 \\ & \cdots & \\ R_n^{(n+1)}(x_0) & = & 0, \\ R_n^{(n+1)}(x) & = & f^{(n+1)}(x). \end{array}.$$

We now apply Cauchy's Mean Value Theorem to R_n, g_n on $[x_0, x]$ to obtain $\xi_1 \in (x_0, x)$, and the MVT to R_n', g_n' on $[x_0, \xi_1]$ and etc,

$$\begin{array}{rcl} \frac{R_n(x)}{g_n(x)} & = & \frac{R_n'(\xi_1)}{g_n'(\xi_1)} = \frac{R_n^{(2)}(\xi_2)}{g_n^{(2)}(\xi_2)} = \cdots = \frac{R_n^{(n)}(\xi_n)}{g_n^{(n)}(\xi_n)} \\ & = & \frac{R_n^{(n+1)}(\xi_{n+1})}{g_n^{(n+1)}(\xi_{n+1})} \\ & = & f^{(n+1)}(\xi_{n+1}). \end{array}$$

where $\xi_k \in (x_0, \xi_{k-1})$, $k = 1, \ldots, n+1$. Taking $\xi = \xi_{n+1}$,

$$R_n(x) = g_n(x) f^{(n+1)}(\xi) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi).$$

$\square$

11.5 A Table

Table of standard Taylor expansions:

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots, && |x| < 1 \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, && \forall x \in \mathbf{R} \\
\log(1+x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, && -1 < x \leq 1 \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, && \forall x \in \mathbf{R} \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, && \forall x \in \mathbf{R} \\
\arctan x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots, && \forall x \in \mathbf{R}, \qquad |x| \leq 1
\end{aligned}$$

Stirling's Formula:

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \ldots).$$

Chapter 12

Techniques for Evaluating Limits

Let c be an extended real number (i.e. $c \in \mathbf{R}$ or $c = \pm\infty$). Suppose that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ are both equal to 0, or both equal to ∞, or both equal to $-\infty$. What can we say about $\lim_{x\to c} f(x)/g(x)$? Which function tend to 0 (respectively $\pm\infty$) faster? We cover two techniques for answering this kind of question: Taylor's Theorem and L'Hôpital's Theorem.

12.1 Use Taylor's Theorem

How do we compute limits using Taylor expansions?

Let $r < s$ be two real numbers and $a \in (r, s)$, and suppose that $f, g \in C^n([r, s])$. Suppose f and g are $n+1$ times differentiable on (r, s). Assume that

$$\begin{aligned} f(a) = f^{(1)}(a) = \dots f^{(n-1)}(a) = 0 \\ g(a) = g^{(1)}(a) = \dots g^{(n-1)}(a) = 0. \end{aligned}$$

Suppose that $f^{(n)}(a) = a_n$, $g^{(n)}(a) = b_n \neq 0$. By Taylor's Theorem,

$$\begin{aligned} f(x) &= \frac{a_n}{n!}(x-a)^n + R_n^f(x) \\ g(x) &= \frac{b_n}{n!}(x-a)^n + R_n^g(x). \end{aligned}$$

Here

$$R_n^f(x) = f^{(n+1)}(\xi_f)\frac{(x-a)^{n+1}}{(n+1)!}, \qquad R_n^g(x) = g^{(n+1)}(\xi_g)\frac{(x-a)^{n+1}}{(n+1)!}$$

for some ξ_f and ξ_g between a and x.

Theorem 12.1 *Suppose that* $f, g \in C^{n+1}([r,s])$, $a \in (r,s)$. *Suppose that*

$$f(a) = f^{(1)}(a) = \ldots f^{(n-1)}(a) = 0,$$
$$g(a) = g^{(1)}(a) = \ldots g^{(n-1)}(a) = 0.$$

Suppose that $g^{(n)}(a) \neq 0$. *Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

Proof Write $g^{(n)}(a) = b_n$ and $g^{(n)}(a) = a_n$. By Taylor's Theorem,

$$\begin{aligned} f(x) &= \frac{a_n}{n!}(x-a)^n + R_n^f(x) \\ g(x) &= \frac{b_n}{n!}(x-a)^n + R_n^g(x). \end{aligned}$$

Here

$$R_n^f(x) = f^{(n+1)}(\xi_f)\frac{(x-a)^{n+1}}{(n+1)!}, \qquad R_n^g(x) = g^{(n+1)}(\xi_g)\frac{(x-a)^{n+1}}{(n+1)!},$$

where $\xi_f, \xi_g \in [r,s]$. If $f^{(n+1)}$ and $g^{(n+1)}$ are continuous, they are bounded on $[r,s]$ (by the Extreme Value Theorem). Then

$$\begin{aligned} \lim_{x \to a} f^{(n+1)}(\xi)\frac{(x-a)}{(n+1)} &= 0, \\ \lim_{x \to a} g^{(n+1)}(\xi)\frac{(x-a)}{(n+1)} &= 0. \end{aligned}$$

So

$$\begin{aligned} \lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{\frac{a_n}{n!}(x-a)^n + R_n^f(x)}{\frac{b_n}{n!}(x-a)^n + R_n^g(x)} \\ &= \lim_{x \to a} \frac{a_n + f^{(n+1)}(\xi)\frac{(x-a)}{(n+1)}}{b_n + g^{(n+1)}(\xi)\frac{(x-a)}{(n+1)}} \\ &= \frac{a_n}{b_n}. \end{aligned}$$

$\square$

Example 12.1

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

Since $\cos x - 1 = -\frac{x^2}{2} + \cdots = 0x - \frac{x^2}{2} + \ldots$.

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \frac{0}{2} = 0.$$

12.2 L'Hôpital's Rule

Question: Suppose that $\lim_{x\to c} \frac{f'(x)}{g'(x)} = \ell$ where $l \in \mathbf{R} \cup \{\pm\infty\}$. Suppose that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ are both equal to 0, or to $\pm\infty$. Can we say something about $\lim_{x\to c} \frac{f(x)}{g(x)}$? We call limits where both f and g tend to 0, "$\frac{0}{0}$-type limits", and limits where both f and g tend to $\pm\infty$ " $\frac{\infty}{\infty}$-type limits". We wiwll deal with both $\frac{0}{0}$- and $\frac{\infty}{\infty}$- type limits. and would like to conclude in both caases that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = \ell.$$

Theorem 12.2 (A simple L'Hôpital's rule) *Let $x_0 \in (a,b)$. Suppose that f, g are differentiable on (a,b). Suppose $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ and $g'(x_0) \neq 0$. Then*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$$

Proof Since $g'(x_0) \neq 0$, there is an interval $(x_0 - r, x_0 + r) \subset (a,b)$ on which $g(x) - g(x_0) \neq 0$. By the algebra of limits, for $x \in (x_0 - r, x_0 + r)$,

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}} = \frac{\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}}{\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}} = \frac{f'(x_0)}{g'(x_0)}.$$

$\square$

Example 12.2 Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$. This is a $\frac{0}{0}$ type limit. Moreover, taking $g(x) = x$, we have $g'(x_0) \neq 0$. So L'Hôpital's rule applies, and

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{\cos x}{1}|_{x=0} = \frac{1}{1} = 1.$$

Example 12.3 Evaluate $\lim_{x\to 0} \frac{x^2}{x^2+\sin x}$.
Since $x^2|_{x=0} = 0$ and $(x^2 + \sin x)|_{x=0} = 0$ we identify this as $\frac{0}{0}$ type. By L'Hôpital's rule,

$$\lim_{x\to 0} \frac{x^2}{x^2 + \sin x} = \frac{2x}{2x + \cos x}|_{x=0} = \frac{0}{0+1} = 0.$$

In the following version of L'Hôpital 's rule, we do not assume that f or g are defined at x_0.

Theorem 12.3 *Let $x_0 \in (a, b)$. Consider $f, g : (a, b)\backslash\{x_0\} \to \mathbf{R}$ and assume that they are differentiable at every point of $(a, b)\backslash\{x_0\}$. Suppose that $g'(x) \neq 0$ for all $x \in (a, b)\backslash\{x_0\}$.*

1. *If $\ell = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists, where ℓ is an extended number, then*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \ell.$$

2. *Suppose that $g'(x) \neq 0$ for all $x \in (a, b)\backslash\{x_0\}$ and*

$$\lim_{x\to x_0} f(x) = \pm\infty, \qquad \lim_{x\to x_0} g(x) = \pm\infty.$$

If $\ell = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \ell.$$

Proof

1. For part 1, we prove only the case where $\ell \in \mathbf{R}$. The case where $\ell = \pm\infty$ is left as an exercise.

   (a) We may assume that f, g are defined and continuous at x_0 and that $f(x_0) = g(x_0) = 0$. For otherwise we simply define

   $$f(x_0) = 0 = \lim_{x\to x_0} f(x), \qquad g(x_0) = 0 = \lim_{x\to x_0} g(x).$$

   (b) Let $x > x_0$. Our functions are continuous on $[x_0, x]$ and differentiable on (x_0, x). Apply Cauchy's Mean Value Theorem to see there exists $\xi \in (x_0, x)$ with

   $$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)}.$$

Hence

$$\lim_{x\to x_0+}\frac{f(x)}{g(x)}=\lim_{x\to x_0+}\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\lim_{x\to x_0+}\frac{f'(\xi)}{g'(\xi)}=\ell.$$

This is because as $x\to x_0$, also $\xi\in(x_0,x)\to x_0$.

(c) Let $x<x_0$. The same 'Cauchy's Mean Value Theorem' argument shows that

$$\lim_{x\to x_0-}\frac{f(x)}{g(x)}=\lim_{x\to x_0-}\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\lim_{x\to x_0-}\frac{f'(\xi)}{g'(\xi)}=\ell.$$

In conclusion $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\ell$.

2. **For part 2.** First the case $\ell=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}\in\mathbf{R}$.

(a) There exists $\delta>0$ such that if $0<|x-x_0|<\delta$,

$$\ell-\epsilon<\frac{f'(x)}{g'(x)}<\ell+\epsilon. \tag{12.1}$$

Since $g'(x)\neq 0$ on $(a,b)\backslash\{x_0\}$, the above expression makes sense.

(b) Take x with $0<|x-x_0|<\delta$, and $y\neq x$ such that $0<|y-x_0|<\delta$, and such that $x-x_0$ and $y-x_0$ have the same sign (so that x_0 does not lie between x and y). Then $g(x)-g(y)\neq 0$ because if $g(x)=g(y)$ there would be some point ξ between x and y with $g'(\xi)=0$, by the Mean Value Theorem, whereas by hypothesis g' is never zero on $(a,b)\smallsetminus\{x_0\}$. Thus

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(y)}{g(x)}+\frac{f(y)}{g(x)}=\left(\frac{f(x)-f(y)}{g(x)-g(y)}\right)\frac{g(x)-g(y)}{g(x)}+\frac{f(y)}{g(x)} \tag{12.2}$$

(c) Fix any such y and let x approach x_0. Since $\lim_{x\to x_0}f(x)=\lim_{x\to x_0}g(x)=+\infty$, we have

$$\lim_{x\to x_0}\frac{f(y)}{f(x)}=0,\qquad \lim_{x\to x_0}\frac{g(x)-g(y)}{g(x)}=1.$$

(exercise)

(d) By Cauchy's Mean Value Theorem,(12.2) can be written

$$\frac{f(x)}{g(x)} = \left(\frac{f'(\xi)}{g'(\xi)}\right)\left(\frac{g(x)-g(y)}{g(x)}\right) + \frac{f(y)}{g(x)} \tag{12.3}$$

for some ξ between x and y. By choosing δ small enough, and x and y as in (b), we can make $\frac{f'(\xi)}{g'(\xi)}$ as close as we wish to ℓ. By taking x close enough to x_0 while keeping y fixed, we can make $\frac{g(x)-g(y)}{g(x)}$ as close as we like to 1, and $\frac{f(y)}{g(x)}$ as close as we wish to 0. It follows from (12.3) that

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \ell.$$

Part 2), case $\ell = +\infty$. For any $A > 0$ there exists $\delta > 0$ such that

$$\frac{f'(x)}{g'(x)} > A$$

whenever $0 < |x - x_0| < \delta$.

Consider the case $x > x_0$. Fix $y > x_0$ with $0 < |y - x_0| < \delta$. By equation (**??**),

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(c)}{g'(c)}$$

for some c between x and y.

Clearly

$$\lim_{x\to x_0} \frac{f(y)}{f(x)} = 0, \qquad \lim_{x\to x_0} \frac{g(x)-g(y)}{g(x)} = 1.$$

It follows (taking $\epsilon = 1/2$ in the definition of limit) that there exists δ_1 such that if $0 < |x - x_0| < \delta_1$,

$$\frac{1}{2} = 1 - \epsilon < \frac{g(x)-g(y)}{g(x)} < 1 + \epsilon = \frac{3}{2}, \qquad -\frac{1}{2} = -\epsilon < \frac{f(y)}{f(x)} < \epsilon = \frac{1}{2}.$$

Thus if $0 < |x - x_0| < \delta$,

$$\begin{aligned} \frac{f(x)}{g(x)} &= \left(\frac{f(x)-f(y)}{g(x)-g(y)}\right)\frac{g(x)-g(y)}{g(x)} + \frac{f(y)}{g(x)} \\ &\geq \frac{A}{2} - \frac{1}{2}. \end{aligned}$$

Since $\frac{A}{2} - \frac{1}{2}$ can be made arbitrarily large we proved that

$$\lim_{x \to x_0+} \frac{f(x)}{g(x)} = +\infty.$$

The proof for the case $\lim_{x \to x_0-} \frac{f(x)}{g(x)} = +\infty$ is similar.

□

Example 12.4 1. Evaluate $\lim_{x\to 0} \frac{\cos x - 1}{x}$.
This is of $\frac{0}{0}$ type.

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{-\sin x}{1} = \frac{0}{1} = 1.$$

2. Evaluate $\lim_{x\to 0} \frac{\arctan(e^x - 1)}{x}$.
The limit $\lim_{x\to 0} \frac{\arctan(e^x-1)}{x}$ is of $\frac{0}{0}$ type, since $e^x - 1 \to 0$ and $\arctan(e^x - 1) \to \arctan(0) = 0$ by the continuity of arctan.

$$\begin{aligned} \lim_{x\to 0} \frac{\arctan(e^x - 1)}{x} &= \lim_{x\to 0} \frac{\frac{1}{1+(e^x-1)^2} e^x}{1} \\ &= \frac{e^0}{1 + (e^0 - 1)^2} = 1. \end{aligned}$$

Example 12.5 Evaluate $\lim_{x\to 1} \frac{e^{2(x-1)}-1}{\ln x}$.
All functions concerned are continuous, $\frac{e^{2(x-1)}-1}{\ln x} = \frac{e^{2(1-1)}-1}{\ln 1} = \frac{0}{0}$.

$$\lim_{x\to 1} \frac{e^{2(x-1)} - 1}{\ln x} = \lim_{x\to 1} \frac{2e^{2(x-1)}}{\frac{1}{x}} = 2.$$

Example 12.6 Don't get carried away with l'Hôpital's rule. Is the following correct and why?

$$\lim_{x\to 2} \frac{\sin x}{x^2} = \lim_{x\to 2} \frac{\cos x}{2x} = \lim_{x\to 2} \frac{-\sin x}{2} = \frac{-\sin 2}{2}?$$

Example 12.7 Evaluate $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$. This is of $\frac{0}{0}$ type.

$$\begin{aligned} \lim_{x\to 0} \frac{\cos x - 1}{x^2} &= \lim_{x\to 0} \frac{-\sin x}{2x} \\ &= \lim_{x\to 0} \frac{-\cos x}{2} \\ &= -\frac{1}{2}. \end{aligned}$$

The limit $\lim_{x\to 0} \frac{-\sin x}{2x}$ is again of $\frac{0}{0}$ type and again we applied L'Hôpital's rule.

Example 12.8 Take $f(x) = \sin(x-1)$ and

$$g(x) = \begin{cases} x-1, & x \neq 1, \\ 0, & x = 1 \end{cases}$$

$$\lim_{x\to 1} f(x) = 0, \lim_{x\to 1} g(x) = 0.$$

$$\lim_{x\to 1} \frac{f'(x)}{g'(x)} = 1.$$

Hence

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = 1.$$

Theorem 12.4 *Suppose that f and g are differentiable on (a, x_0) and $g(x) \neq 0$, $g'(x) \neq 0$ for all $x \in (a, x_0)$. Suppose*

$$\lim_{x\to x_0-} f(x) = \lim_{x\to x_0-} g(x)$$

is either 0 or infinity. Then

$$\lim_{x\to x_0-} \frac{f(x)}{g(x)} = \lim_{x\to x_0-} \frac{f'(x)}{g'(x)}.$$

provided that $\lim_{x\to x_0-} \frac{f'(x)}{g'(x)}$ exists.

Example 12.9 Evaluate $\lim_{x\to 0+} x \ln x$.
Since $\lim_{x\to 0+} \ln x = -\infty$ we identify this as a "$0 \cdot \infty$- type" limit. We have not studied such limits, but can transform it into a $\frac{\infty}{\infty}$-type limit, which we know how to deal with:

$$\lim_{x\to 0+} x \ln x = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} (-x) = 0.$$

Exercise 12.10 Evaluate $\lim_{y\to\infty} \frac{-\ln y}{y}$.

Theorem 12.5 *Suppose f and g are differentiable on (a, ∞) and $g(x) \neq 0$, $g'(x) \neq 0$ for all x sufficiently large.*

1. *Suppose*

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = 0$$

Then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

provided that $\lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$ *exists.*

2. *Suppose*

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = \infty$$

Then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

provided that $\lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$ *exists.*

Proof Idea of proof: Let $t = \frac{1}{x}$, $F(t) = f(\frac{1}{t})$ and $G(t) = g(\frac{1}{t})$

$$\lim_{x\to+\infty} f(x) = \lim_{t\to0} f(\frac{1}{t}) = \lim_{t\to0} F(t).$$

$$\lim_{x\to+\infty} g(x) = \lim_{t\to0} g(\frac{1}{t}) = \lim_{t\to0} G(t).$$

Also

$$\frac{F'(t)}{G'(t)} = \frac{f'(\frac{1}{t})(-\frac{1}{t^2})}{g'(\frac{1}{t})(-\frac{1}{t^2})} = \frac{f'(x)}{g'(x)}.$$

Note that the limit as $x \to \infty$ is into a limit as $t \to)^+$, which we do know how to deal with. □

Example 12.11 Evaluate $\lim_{x\to\infty} \frac{x^2}{e^x}$.
We identify this as $\frac{\infty}{\infty}$ type.

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

We have to apply L'Hôpital's rule twice since the result after applying L'Hôpital's rule the first time is $\lim_{x\to\infty} \frac{2x}{e^x}$, which is again $\frac{\infty}{\infty}$ type.

Proposition 12.6 *Suppose that* $f, g : (a, b) \to \mathbf{R}$ *are differentiable, and* $f(c) = g(c) = 0$ *for some* $c \in (a, b)$. *Suppose that* $\lim_{x\to c} \frac{f'(x)}{g'(x)} = +\infty$. *Then* $\lim_{x\to c} \frac{f(x)}{g(x)} = +\infty$

Proof For any $M > 0$ there is $\delta > 0$ such that if $0 < |x - c| < \delta$,

$$\frac{f'(x)}{g'(x)} > M.$$

First take $x \in (c, c + \delta)$. By Cauchy's mean value theorem applied to the interval $[c, x]$, there exists ξ with $|\xi - c| < \delta$ such that

$$\begin{aligned} \frac{f(x)}{g(x)} &= \frac{f(x) - f(c)}{f(x) - g(c)} \\ &= \frac{f'(\xi)}{g'(\xi)} > M. \end{aligned}$$

hence $\lim_{x \to x_0+} \frac{f(x)}{g(x)} = +\infty$. A similar argument shows that $\lim_{x \to x_0-} \frac{f(x)}{g(x)} = +\infty$ □

Remark 12.1 We summarise the cases: Let x_0 be an extended real number and I be an open interval which either contains x_0 or has x_0 as an end point. Suppose f and g are differentiable on I and $g(x) \neq 0$, $g'(x) \neq 0$ for all $x \in I$. Suppose

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$$

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x_0)}{g'(x_0)}.$$

provided that $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists. Suitable one sided limits are used if x_0 is an end-point of the interval I.

Recall that if α is a number, for $x > 0$ we define

$$x^\alpha = e^{\alpha \log x}.$$

So

$$\frac{d}{dx} x^\alpha = e^{\alpha \log x} \frac{\alpha}{x} = x^\alpha \frac{\alpha}{x} = \alpha x^{\alpha - 1}.$$

Example 12.12 For $\alpha > 0$,

$$\lim_{x \to \infty} \frac{\log x}{x^\alpha} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\alpha x^{\alpha - 1}} = \lim_{x \to \infty} \frac{1}{\alpha x^\alpha} = 0.$$

Conclusion: As $x \to +\infty$, $\log x$ goes to infinity slower than x^α any $\alpha > 0$.

Example 12.13 Evaluate $\lim_{x\to 0}(\frac{1}{x} - \frac{1}{\sin x})$.
This is apparently of "$\infty - \infty$-type". We have

$$\begin{aligned}
\lim_{x\to 0}(\frac{1}{x} - \frac{1}{\sin x}) &= \lim_{x\to 0} \frac{\sin x - x}{x \sin x} \\
&= \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} \\
&= \lim_{x\to 0} \frac{-\sin x}{2\cos x - x\sin x} \\
&= \frac{-0}{2-0} = 0.
\end{aligned}$$

Example 12.14 Evaluate $\lim_{x\to 0}(\frac{1}{x} - \frac{1}{\sin x})$.

We try Taylor's method. For some ξ, $\sin x = x - \frac{x^3}{3!}\cos\xi$:

$$\begin{aligned}
\lim_{x\to 0}(\frac{1}{x} - \frac{1}{\sin x}) &= \lim_{x\to 0} \frac{\sin x - x}{x \sin x} \\
&= \lim_{x\to 0} \frac{-\frac{x^3}{3!}\cos\xi}{x(x - \frac{x^3}{3!}\cos\xi)} \\
&= \lim_{x\to 0} \frac{-\frac{x}{3!}\cos\xi}{(1 - \frac{x}{3!}\cos\xi)} = 0.
\end{aligned}$$

Chapter 13

Unfinished Business

Theorem 13.1 (The Inverse Function Theorem) *Suppose that $f : (a, b) \to \mathbf{R}$ is of type C^k, for some $k \in \mathbf{N} \cup \{\infty\}$, and $x_0 \in (a, b)$.*

1. *If $f'(x_0) > 0$, then there exists $r > 0$ such that $(x_0 - r, x_0 + r) \subset (a, b)$ and such that $f : (x_0 - r, x_0 + r) \to (f(x_0 - r), f(x_0 + r))$ is a bijection with inverse of type C^k.*

2. *If $f'(x_0) < 0$, then there exists $r > 0$ such that $(x_0 - r, x_0 + r) \subset (a, b)$ and such that $f : (x_0 - r, x_0 + r) \to (f(x_0 + r), f(x_0 - r))$ is a bijection with inverse of type C^k.*

Proof We only consider the case $f'(x_0) > 0$. Since f' is continuous there exists $r > 0$ such that $f'(x) > 0$ for all $x \in (x_0 - r, x_0 + r)$. The function f is increasing on $[x_0 - r, x_0 + r]$, by Corollary 7.5. Hence, by the Intermediate Value Theorem, f is a continuous bijection from $(x_0 - r, x_0 + r)$ to $(f(x_0 - r), f(x_0 + r))$. [Comment: It has a continuous inverse by the continuous version of the Inverse Function Theorem (Theorem 5.3).] By the differential version of the Inverse Function Theorem (Thm. 6.7), f^{-1} is differentiable at any $y \in (f(x_0 - r), f(x_0 + r))$ and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

Since f' is continuous and f^{-1} is continuous, the composition $(f^{-1})'$ of the two continuous functions is continuous. This proves the case $k = 1$ of the theorem.

If $k \geq 2$ this we use induction. Suppose that we have proved that if f is of type $C^{(k-1)}$ then f^{-1} is of type $C^{(k-1)}$. We now show that this holds for

k. If f is $C^{(k)}$, then since $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$, we see that $(f^{-1})'$ is the composition of two $C^{(k-1)}$ functions and is therefore $C^{(k-1)}$. Thus $(f^{-1})'$ is $C^{(k-1)}$ which means that f^{-1} is $C^{(k)}$. $\square$